

# BOOLEAN ULTRAPOWERS

by

Washiela Fish

A thesis prepared under the supervision of Dr. H. Rose in fulfilment of the requirements for the degree of Master of Science in Mathematics

UNIVERSITY OF CAPE TOWN

Department of Mathematics  
and Applied Mathematics

Copyright by the University of Cape Town  
November 2000

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.

# Synopsis

The Boolean ultrapower construction is a generalisation of the ordinary ultrapower construction in that an arbitrary complete Boolean algebra replaces the customary powerset Boolean algebra. B. Koppelberg and S. Koppelberg [1976] show that the class of ordinary ultrapowers is properly contained in the class of Boolean ultrapowers thereby justifying the development of a theory for Boolean ultrapowers. This thesis is an exploration into the strategies whereby and the conditions under which aspects of the theory of ordinary ultrapowers can be extended to the theory of Boolean ultrapowers. Mansfield [1971] shows that a finitely iterated Boolean ultrapower is isomorphic to a single Boolean ultrapower under certain conditions. Using a different approach and under somewhat different conditions, Ouwehand and Rose [1998] show that the result also holds for  $\kappa$ -bounded Boolean ultrapowers. Mansfield [1971] also proves a Boolean version of the Keisler-Shelah theorem. By redefining the notion of a  $\kappa$ -good ultrafilter on a Boolean algebra, Benda [1974] obtains a complete generalisation of a theorem of Keisler which states that an ultrapower is  $\kappa$ -saturated iff the ultrafilter is  $\kappa$ -good. Potthoff [1974] defines the notion of a limit Boolean ultrapower and shows that, as is the case for ordinary ultrapowers, the complete extensions of a model are characterised by its limit Boolean ultrapowers. Upon the discovery by Frayne, Morel and Scott [1962] of an ultrapower of a simple group which is not simple, Burris and Jeffers [1978] investigate necessary and sufficient conditions for a Boolean ultrapower to be simple, or subdirectly irreducible, provided that the language is countable. Finally, Jipsen, Pinus and Rose [2000] extend the notion of the Rudin-Keisler ordering to ultrafilters on complete Boolean algebras, and prove that by using this definition, Blass' Characterisation Theorem can be generalised for Boolean ultrapowers.

## Acknowledgements

This thesis was written at the University of Cape Town under the supervision of Dr. Henry Rose. I wish to express my sincerest gratitude to Dr. Rose for agreeing to have me as a student after a period of absence of twelve years from tertiary-level mathematics. Dr. Rose has taught me during my undergraduate years and encouraged me to pursue further studies in mathematics. I have a deep respect for his tireless dedication to the subject and his uncompromising stance towards standards in the learning of mathematics. In fact, his approach to the teaching and learning of mathematics has provided me with valuable insights into the current problems in mathematics education in South Africa. I thank him for his patience during the times that I have had to overcome personal hurdles.

I also wish to thank the National Science Foundation for financial support during my year of full-time study.

Finally, I wish to thank my family and my colleagues at the Alternative Admissions Research Project at UCT for their support and encouragement.

# Table of Contents

<b>Synopsis</b>	i
<b>Acknowledgements</b>	ii
<b>Introduction</b>	v
<b>Summary</b>	viii

## Chapter 0: Preliminary Results

0.1 Models and Boolean algebras	1
0.2 The various constructions of $M[B]$ and $M[B]_\omega$	9
0.2.1 A.L. Foster [1953]	10
0.2.2 B. Banaschewski and E. Nelson [1980]	10
0.2.3 M. Benda [1974]	16
0.2.4 P. Ouwehand and H. Rose [1998]	17
0.3 Basic algebraic aspects of Boolean powers	23
0.4 Model-theoretic aspects of Boolean powers	29
0.5 Boolean ultrapowers	40
0.6 A diversion : Los's Theorem for the $(B, P)$ -ultraproduct of models	44

## Chapter 1: Boolean ultrapowers which are not isomorphic to ordinary ultrapowers

1.1 The original construction of B. Koppelberg and S. Koppelberg [1976] of a Boolean ultrapower which is not isomorphic to any ordinary ultrapower	48
1.2 The construction of B. Koppelberg [1980] of Boolean ultrapowers of $\langle \omega, < \rangle$ and $\langle \omega_1, < \rangle$ which are not isomorphic to any ordinary ultrapowers	53

## Chapter 2: Finitely iterated Boolean ultrapowers

2.1 The approach of R. Mansfield [1971] to the problem of finitely iterated Boolean ultrapowers	59
2.2 The approach of P. Ouwehand and H. Rose [1998] to the problem of finitely iterated Boolean ultrapowers	67

**Chapter 3: Elementarily equivalent models have isomorphic Boolean ultrapowers** 74

**Chapter 4: Good ultrafilters and saturated Boolean ultrapowers**

- 4.1 The notion of goodness of an ultrafilter as defined by R. Mansfield [1971] and his partial generalisation of Keisler's theorem 80
- 4.2 The notion of goodness of an ultrafilter as defined by M. Benda [1974] and his complete generalisation of Keisler's theorem 85

**Chapter 5: Limit Boolean Ultrapowers**

- 5.1 The characterisation of the full submodels of a Boolean power of  $K$ . Potthoff [1974] 89
- 5.2 The definition of a limit Boolean ultrapower of  $K$ . Potthoff [1974] and his results concerning limit Boolean ultrapowers 95

**Chapter 6: Simple and subdirectly irreducible Boolean ultrapowers** 100

**Chapter 7: Rudin-Keisler posets of complete Boolean algebras**

- 7.1 The extension of the Rudin-Keisler ordering to ultrafilters on complete Boolean algebras of P. Jipsen et al [2000] 103
- 7.2 The characterisation of the Rudin-Keisler ordering by elementary embeddings between Boolean ultrapowers 106
- 7.3 The conditions under which the Rudin-Keisler poset of a Boolean algebra is order-embeddable into the Rudin-Keisler poset of another Boolean algebra 110

**List of Symbols** 114

**Index** 116

**References** 121

## Introduction

From a model  $M$  of the language  $L$  and a Boolean algebra  $B$  can be obtained a new model  $M[B]$  called the Boolean power of  $M$  by  $B$  which is of the same type as  $M$ . Hodges [1993] remarks that “Boolean powers pass the first test of a good construction: they are used to prove interesting theorems which don’t mention them”. Boolean powers are important tools for proving theorems about decidability,  $\aleph_0$ -categoricity and the existence of many non-isomorphic models. In his survey of Boolean constructions, Burris [1982] commented that the Boolean power construction resulted in extensive undecidability theorems.

According to Burris’ [1982] account, Boolean powers were first introduced by R.F. Arens and I. Kaplansky in 1948. They incorporated the ideas of M.H. Stone and I. Gelfand. Stone had shown that a Boolean ring with unit is the algebra of clopen subsets of a compact zero-dimensional space. Put differently, Stone’s result asserts that a Boolean ring with unit is the algebra of continuous functions from a compact zero-dimensional space to the 2-element field. Gelfand had shown that certain Banach algebras are isomorphic to the algebra of continuous functions from a compact Hausdorff space to the set of real or complex numbers. Arens and Kaplansky investigated representations of the algebra of continuous functions from a compact zero-dimensional space to a simple ring which had been given the discrete topology, and generalisations of such representations. Thus they introduced the definitions of the bounded Boolean power  $M[B]_\omega$  in the case that  $M$  is a ring. Hodges [1993] mentions that in 1959, Philip Hall defined the bounded Boolean power  $M[B]_\omega$  in the case that  $M$  is a group. A.L. Foster [1953], who was working completely independently of Arens and Kaplansky, presented an alternative definition of the Boolean power  $M[B]$  which had its roots in algebra as opposed to topology. Foster defined the Boolean power  $M[B]$  in the case that  $M$  is an algebra. (The generalised definition of the Boolean power  $M[B]$  that is used extensively requires only that  $M$  is a model of a first-order language  $L$ .) This definition, however, had one drawback, namely that if  $M$  is infinite, then  $B$  is required to be a complete Boolean algebra. To get around this requirement, Foster introduced the notion of the bounded Boolean power  $M[B]_\omega$  in 1961 by adding to the definition of  $M[B]$  the condition that only finitely many elements of  $M$  have non-zero images in  $B$ . In his review of Foster’s paper of 1961, B. Jónsson observed and commented on the natural isomorphism which existed between the bounded Boolean power  $M[B]_\omega$  and the algebra of continuous functions from the Stone space of  $B$  to the algebra  $M$  which had been given the discrete topology. By restricting their attention to those functions that are continuous on dense open subsets of the Stone space of  $B$  (and in the process complicating the above situation somewhat), B. Banaschewski and E. Nelson [1980] gave an analogous definition of the Boolean power  $M[B]$ . With

reference to the topological description of  $M[B]_\omega$  and  $M[B]$ , Banaschewski and Nelson [1980] lament the fact that “these natural and convenient descriptions of Boolean powers have never actually been used as a basis for their study”, and the objective of their paper was the presentation of a systematic study of Boolean powers using a topological framework. Whether this approach actually resulted in the simplification of the proofs of those results that were already known or “placed the subject into a much more conceptual framework” as they claimed, is questionable. However, they did show that  $M[B]$  preserves elementary equivalence and elementary embeddings in both arguments, that  $M[B]_\omega$  is elementarily equivalent to  $M[B]$ , and that the canonical embedding of  $M[B]_\omega$  into  $M[B]$  is elementary.

C.J. Ash [1975] defined a more general version of the Boolean power  $M[B]$  as originally introduced by Foster by weakening the condition that  $B$  be a complete Boolean algebra. For an infinite cardinal  $\kappa$ , he defined the  $\kappa$ -restricted Boolean extension  $M[B]_\kappa$  of  $M$  by stipulating that  $B$  be a  $\kappa$ -complete Boolean algebra and that, in a similar way as for bounded Boolean powers, fewer than  $\kappa$  elements of  $M$  have non-zero images in  $B$ . P. Ouwehand and H. Rose [1998] adopted a novel approach to investigating the model  $M[B]_\kappa$  which they subsequently termed the  $\kappa$ -bounded Boolean power. They observed that any  $\kappa$ -complete Boolean algebra  $B$  is the direct limit of an updirected set of powerset Boolean algebras and proceeded to construct  $M[B]_\kappa$  as the direct limit of an updirected set of direct powers  $M^P$  where  $P$  is a partition of  $B$  consisting of fewer than  $\kappa$  elements. Using this approach, it is easily noted that  $M$  is elementarily embeddable into  $M[B]_\kappa$  and that if  $B$  is isomorphic to some powerset Boolean algebra, then  $M[B]_\kappa$  reduces to an ordinary direct power. By applying Feferman-Vaught techniques, Ouwehand and Rose showed that if  $\kappa \leq \lambda$ , then  $M[B]_\kappa$  is an elementary submodel of  $M[B]_\lambda$ , and by applying a result of Tarski referred to in Monk and Bonnet [1989], also showed that all infinite direct powers of a model are elementarily equivalent.

When the Boolean power  $M[B]$  is factored by a filter  $F$  on  $B$ , then the resulting model  $M[B]/F$  is referred to as a filtral power.  $M[B]/F$  is a  $B/F$ -valued model and is of the same type as  $M$ . In the case that  $F$  is an ultrafilter,  $M[B]/F$  is called a Boolean ultrapower. Boolean ultrapowers are at the heart of this thesis. In the same way that the notion of a Boolean power is a generalisation of the notion of a direct power, the notion of a Boolean ultrapower is a generalisation of the notion of an ordinary ultrapower. R. Mansfield [1971] confidently remarks that he does “not know of a single proof about ultrapowers which cannot be translated directly to a corresponding proof about Boolean ultrapowers”. In fact, the greater flexibility of the more general notion allowed for the construction of isomorphic Boolean ultrapowers, good ultrafilters and saturated Boolean ultrapowers independently of the Generalised Continuum Hypothesis.



Although Boolean ultrapowers are acknowledged as extremely useful tools in Model Theory, the objective of this thesis is not to explore the extent to which Boolean ultrapowers can be used, but to give a comprehensive account of Boolean ultrapowers in their own right. In this regard, the work of authors such as Mansfield [1971], Burris [1975], Koppelberg and Koppelberg [1976], Benda [1974], Potthoff [1974] and Ouwehand and Rose [1998] will be given extensive coverage.

From Chapter 0 it will become abundantly clear that there are different, equally valid ways of constructing a Boolean ultrapower which have implications for the notation and the method of proof used. Although the treatment of Boolean ultrapowers is entirely algebraic in this thesis, it has unfortunately not been possible to devise a uniform system of notation, nor has it been possible to use the same construction in all the proofs. In each section it will be made clear from the outset which construction will be used. Note however, that in certain chapters the use of more than one construction has been inevitable, and the reader is requested to bear this in mind throughout the thesis.

The basic notions from the theory of Boolean algebras referred to in this thesis can be found in the Handbook of Boolean Algebras (ed. Monk and Bonnet [1989]), those from Model Theory in Bell and Slomson [1971], Chang and Keisler [1973] and Hodges [1993], those from Set Theory in Jech [1978], and those from Universal Algebra in Burris and Sankappanavar [1981].

## Summary

In Chapter 0 the basic notions which are relevant to models and Boolean algebras are discussed. The construction of  $M[B]$  and  $M[B]_\omega$  as proposed by Foster [1953], Banaschewski and Nelson [1980], Benda [1974] and Ouwehand and Rose [1998], as well as the two constructions given in Hodges [1993], are discussed in great detail, and the models resulting from these constructions are shown to be isomorphic. The basic algebraic properties and the model-theoretic aspects of Boolean powers are presented as a basis for the results in later chapters. The Boolean ultrapower  $M[B]/F$  is constructed by defining a suitable equivalence relation w.r.t.  $F$  on  $M[B]$ . The construction of the more general  $(B, P)$ -ultraproduct of models is included. Los's Theorem as formulated for the  $(B, P)$ -ultraproducts of models, is shown to coincide with the version for ordinary ultraproducts when each Boolean algebra is isomorphic to the 2-element Boolean algebra, and with the version for Boolean ultrapowers when the models are identical.

In Chapter 1 the construction of B. Koppelberg and S. Koppelberg [1976] of a Boolean ultrapower that is not isomorphic to any ordinary ultrapower is discussed in detail. This construction finally put an end to speculation by Mansfield [1970] amongst others, about the existence of such a Boolean ultrapower. However, the cardinalities of both the language and the model were large. B. Koppelberg [1980], upon noting the construction of W. Lange of such a Boolean ultrapower of which the language has cardinality  $\aleph_1$ , and the model has cardinality  $2^{\aleph_1}$ , constructs Boolean ultrapowers of  $\langle \omega, < \rangle$  and  $\langle \omega_1, < \rangle$  which are not isomorphic to any ordinary ultrapower. His constructions, however, require very strong set-theoretical assumptions. The essence of Koppelberg's constructions is the investigation into Boolean ultrapowers of  $\langle \omega, < \rangle$  and  $\langle \omega_1, < \rangle$  which have cardinalities and cofinalities which are not the cardinalities and cofinalities of any ultrapower.

In Chapter 2 the approaches of Ouwehand and Rose [1998] and Mansfield [1971] to the problem of the finite iteration of Boolean ultrapowers is discussed. Mansfield shows that if two complete Boolean algebras  $A$  and  $B$  are such that  $B$  is  $(\kappa, \infty)$ -distributive and  $A$  satisfies the  $\kappa$ -chain condition w.r.t.  $B$ , then the iterated Boolean ultrapower  $(M[A]/F)[B]/G$  is isomorphic to a single Boolean ultrapower. Ouwehand and Rose show that the same result holds for  $\kappa$ -bounded Boolean ultrapowers on condition that  $A$  is  $\kappa$ -partition complete. If both  $A$  and  $B$  are  $\kappa$ -partition complete, then by a result of Jipsen and Rose [1999],  $A$  and  $B$  are  $\kappa$ -complete and hence the result of Ouwehand and Rose holds. Also, if  $A$  is the completion of a  $(\kappa, \kappa)$ -tree and  $\kappa$  is strongly inaccessible, then since  $\mathbb{P}_A$  is

a  $\kappa$ -complete lattice and  $\mathbb{P}_A^\kappa = \mathbb{P}_A$ , by another result of Jipsen and Rose [1999],  $A$  is  $\kappa$ -partition complete and the result of Ouwehand and Rose holds.

In Chapter 3 Mansfield's Boolean version of the Keisler-Shelah Theorem is presented. Mansfield shows that if any two  $\aleph_1$ -saturated models are elementarily equivalent, then they have isomorphic Boolean ultrapowers. He constructs an inner product into an appropriate Boolean algebra and shows that the models have isomorphic Boolean powers, and hence they have isomorphic Boolean ultrapowers. By applying the Finite Iteration Theorem and the result which states that a Boolean ultrapower is  $\aleph_1$ -saturated if the ultrafilter is descendingly countably incomplete, Mansfield is able to drop the condition that the models be  $\aleph_1$ -saturated and hence obtains the required generalisation.

In Chapter 4 Mansfield's one-directional generalisation of Keisler's theorem which states that an ultrapower is  $\kappa$ -saturated iff the ultrafilter is  $\kappa$ -good is discussed. Benda [1974] notes that Mansfield's definition of a  $\kappa$ -good ultrafilter is not appropriate for obtaining the converse of Mansfield's result. He thus proposes another notion of a  $\kappa$ -good ultrafilter to obtain the complete generalisation of Keisler's theorem. This notion coincides with the usual notion if the Boolean algebra is atomic, and is weaker if it is atomless.

In Chapter 5 Potthoff's [1974] investigation into the full submodels of a Boolean power is given. This investigation culminates in his definition of the notion of a limit Boolean ultrapower. Potthoff defines a limit Boolean ultrapower as a submodel of a Boolean ultrapower of which the ranges of the elements are included in the members of a family of regular subalgebras directed by inclusion. By using this definition, Potthoff shows that the complete extensions of a model are characterised by its limit Boolean ultrapowers.

In Chapter 6 Burris and Jeffers [1978] investigate necessary and sufficient conditions for a Boolean ultrapower to be simple (subdirectly irreducible). They define the notion of a simplicity (subdirect irreducibility) sentence and show that an  $\aleph_0$ -saturated model is simple (subdirectly irreducible) iff it satisfies a simplicity (subdirect irreducibility) sentence. By applying the result which states that a Boolean ultrapower  $M[B]/F$  is  $\aleph_1$ -saturated (and hence  $\aleph_0$ -saturated) if  $F$  is descendingly countably incomplete, they conclude that  $M[B]/F$  is simple (subdirectly irreducible) iff  $M$  satisfies a simplicity (subdirect irreducibility) sentence.

In Chapter 7 details are provided about the Rudin-Keisler ordering of ultrafilters on complete Boolean algebras as defined by Jipsen, Pinus and Rose [2000]. This definition coincides with the usual Rudin-Keisler ordering if the Boolean algebra is a powerset Boolean algebra. This definition facilitates the generalisation

of Blass' Theorem which states that the usual Rudin-Keisler ordering is characterised by elementary embeddings between ordinary ultrapowers. They also investigate the conditions under which the Rudin-Keisler poset of a Boolean algebra is order-embeddable into the Rudin-Keisler poset of another Boolean algebra.

# Chapter 0

## Preliminary Results

Because the universe of the Boolean power  $M[B]$  is a special set of functions from the model  $M$  into the Boolean algebra  $B$  (or vice versa, as in the constructions of Benda [1974] and Ouwehand and Rose [1998]), the theory of Boolean powers, and hence that of Boolean ultrapowers is a fusion of the theories of models in general and Boolean algebras in particular. For example, the images of the above-mentioned functions are elements of a Boolean algebra and hence the properties of the Boolean algebra determine the constraints on these elements. As another example of the interrelatedness of the above-mentioned theories, the validity of a formula in a model determines the validity of a formula in its Boolean ultrapowers.

In this chapter then, a short description of models and Boolean algebras is deemed useful. Thereafter, the various constructions of  $M[B]$  and  $M[B]_\omega$  will be given, not just for the sake of completeness, but to establish the premises on which authors such as Mansfield [1971], Benda [1974] and Ouwehand and Rose [1998] based their results. Although the topological constructions of  $M[B]$  and  $M[B]_\omega$  will not be referred to again in this thesis, their inclusion adds another dimension to the topic. The basic algebraic properties and the model-theoretic aspects of Boolean powers are discussed in detail to provide a foundation for the development of results in later chapters. This chapter is concluded with the construction of the Boolean ultrapower  $M[B]/F$  which is shown to be an elementary extension of  $M$ . A diversion is provided by the construction of the more general  $(B, P)$ -ultraproduct of models, and the formulation of Los's Theorem for  $(B, P)$ -ultraproducts, which is shown to be equivalent to the ordinary version when each Boolean algebra is isomorphic to the 2-element Boolean algebra, and to the version for Boolean ultrapowers when the models are identical.

### 0.1 Models and Boolean algebras

A Boolean algebra is just a special kind of model, and therefore the results which are valid for models in general, are valid for Boolean algebras in particular.

The following basic model-theoretic concepts can be found in Bell and Slomson [1971].

**Definition 0.1.1:**

Suppose that  $L$  is a first order language. A *model* is a tuple  $\langle M, \{R_\xi : \xi < \alpha\}, \{H_\delta : \delta < \beta\} \rangle$  such that  $M$  is a set, and for each  $\xi < \alpha$  and each  $\delta < \beta$ , the relation  $R_\xi$  and the function  $H_\delta$  on  $M$  are the interpretations of the relation symbol  $\overline{R}_\xi$  and the function symbol  $\overline{H}_\delta$  respectively in  $L$ .

To cut down on the use of complex notation which does not add to the significance of the notions involved,  $M$  will be used to represent the model  $\langle M, \{R_\xi : \xi < \alpha\}, \{H_\delta : \delta < \beta\} \rangle$  in this thesis.

**Definition 0.1.2 :**

Suppose that  $M$  is a model of the language  $L$ , and that  $\mu$  and  $\nu$  are functions such that  $\mu : \alpha \rightarrow \omega$  and  $\nu : \beta \rightarrow \omega$ . If for each  $\xi < \alpha$  and each  $\delta < \beta$ ,  $R_\xi$  is a  $\mu(\xi)$ -ary relation and  $H_\delta$  is a  $\nu(\delta)$ -ary function on  $M$ , then  $M$  is a model of *type*  $(\mu, \nu)$ .

As will become evident later in this chapter, each relation and each function on  $M[B]$  is the extension of the corresponding relation and function respectively on  $M$ , and hence  $M$  and  $M[B]$  are models of the same type.

Various relations can hold between models of the same type, and these are considered below.

**Definition 0.1.3 :**

Suppose that  $M$  and  $N$  are models of the language  $L$ . If  $M \subseteq N$ , and each relation and each function on  $M$  is the restriction of the corresponding relation and function respectively on  $N$  to  $M$ , then  $M$  is a *submodel* of  $N$ .  $N$ , in turn, is an *extension* of  $M$ .

For the sake of simplicity, a relation on  $M$  and its corresponding relation on  $N$  will be denoted by the same symbol. A similar remark applies to functions on  $M$ .

**Definition 0.1.4 :**

Suppose that  $M$  and  $N$  are models of the language  $L$ . A function  $h : M \rightarrow N$  is a *homomorphism* if for each relation  $R$  on  $M$  and each  $m_1, \dots, m_n \in M$ ,

$$M \models R(m_1, \dots, m_n) \text{ implies } N \models R(h(m_1), \dots, h(m_n)).$$

The same holds for each function  $H$  on  $M$ . If, in addition,  $h$  is an embedding of  $M$  onto  $N$  and its inverse is also a homomorphism, then  $M$  is *isomorphic* to  $N$ : in symbols,  $M \cong N$ .

The isomorphism relation between models is a purely algebraic relation which is independent of the language  $L$ . The following relation between models involves the language in a fundamental way.

**Definition 0.1.5 :**

Suppose that  $M$  and  $N$  are models of the language  $L$ . If for each sentence  $\phi$  in  $L$ ,

$$M \models \phi \text{ iff } N \models \phi,$$

then  $M$  is *elementarily equivalent* to  $N$ : in symbols,  $M \equiv N$ .

The properties of models which can be described by sentences in  $L$  are of a simple or elementary (in its English usage) nature, and hence the term *elementarily equivalent* is used for the notion introduced above. If any two models are isomorphic, then they are elementarily equivalent. However, the converse is false because there may be no sentence in  $L$  which distinguishes between the two models. For example, all infinite direct powers of a model are clearly not isomorphic, but it will be shown in Corollary 0.4.16 that they are elementarily equivalent.

The following two relations are stronger than the elementarily equivalence relation in that they involve all formulas in  $L$ .

**Definition 0.1.6 :**

Suppose that  $M$  and  $N$  are models of the language  $L$ , and that  $M$  is a submodel of  $N$ . If for each formula  $\phi(x_1, \dots, x_n)$  in  $L$  and each  $m_1, \dots, m_n \in M$ ,

$$M \models \phi(m_1, \dots, m_n) \text{ iff } N \models \phi(m_1, \dots, m_n),$$

then  $M$  is an *elementary submodel* of  $N$ : in symbols,  $M \prec N$ .  $N$ , in turn, is an *elementary extension* of  $M$ .

**Definition 0.1.7 :**

Suppose that  $M$  and  $N$  are models of the language  $L$ . An embedding  $e : M \rightarrow N$  is an *elementary embedding* if for each formula  $\phi(x_1, \dots, x_n)$  in  $L$  and each  $m_1, \dots, m_n \in M$ ,

$$M \models \phi(m_1, \dots, m_n) \text{ iff } N \models \phi(e(m_1), \dots, e(m_n)).$$

If there exists an elementary embedding from  $M$  into  $N$ , then  $M$  is *elementarily embeddable* in  $N$ .

The following lemma, which is known as the Tarski-Vaught Criterion, provides a very useful criterion for determining whether a submodel is in fact an elementary submodel.

**Lemma 0.1.8 [Tarski-Vaught Criterion]:**

*Suppose that  $M$  and  $N$  are models of the language  $L$ , and that  $M$  is a submodel of  $N$ . Then  $M$  is an elementary submodel of  $N$  iff for each formula  $\phi$  in  $L$  and each  $m_1, \dots, m_n \in M$  such that  $N \models \exists x \phi(x, m_1, \dots, m_n)$ , there exists  $\tilde{m} \in M$  such that  $N \models \phi(\tilde{m}, m_1, \dots, m_n)$ .*

The following basic algebraic concepts can be found in Burris and Sankappanavar [1981], Monk and Bonnet [1989] and Bell and Slomson [1971].

**Definition 0.1.9 :**

Suppose that  $L$  is a language consisting of the finitary function symbols  $\{\overline{H}_\delta : \delta < \beta\}$ . An *algebra* is a model  $\langle A, \{H_\delta : \delta < \beta\} \rangle$  such that for each  $\delta < \beta$ , the function  $H_\delta$  on  $A$  is the interpretation of the corresponding function symbol  $\overline{H}_\delta$  in  $L$ .

**Definition 0.1.10 :**

A *Boolean algebra* is a model  $\langle B, \{\vee, \wedge, {}^c\}, \{0, 1\} \rangle$  such that  $\vee, \wedge$  are binary operations,  ${}^c$  is a unary operation, and  $0, 1$  are distinguished elements such that for all  $b_1, b_2, b_3 \in B$ ,

- (a)  $b_1 \vee (b_2 \vee b_3) = (b_1 \vee b_2) \vee b_3$ , (associativity)
- (b)  $b_1 \vee b_2 = b_2 \vee b_1$ , (commutativity)
- (c)  $b_1 \vee (b_1 \wedge b_2) = b_1$ , (absorption)
- (d)  $b_1 \wedge (b_2 \vee b_3) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$ , (distributivity)
- (e)  $b_1 \vee b_1^c = 1$ ,  $b_1 \wedge b_1^c = 0$ . (complementation)

By defining the relation  $\leq$  on  $B$  by

$$b_1 \leq b_2 \text{ iff } b_1 \vee b_2 = b_2,$$

$\langle B, \leq \rangle$  becomes a partial order in which the least upper bound and the greatest lower bound of  $\{b_1, b_2\}$  are  $b_1 \vee b_2$  and  $b_1 \wedge b_2$  respectively.

Boolean algebras are named after the English mathematician George Boole who, in his book “The Mathematical Analysis of Logic, being an essay towards a calculus of deductive reasoning”, applied mathematical techniques to logic. Although the notion of a Boolean algebra was developed early in the 19th century through working with operations such as union, intersection and complementation, the abstract notion first became evident in the work of E.V. Huntington in 1904



(recorded in Burris [1982]). Huntington is credited with the development of the system of axioms (a) to (e) above, and the establishment of the equivalence between complemented, distributive lattices and Boolean algebras.

**Example 0.1.11**

The powerset  $\mathcal{P}(X)$  becomes a Boolean algebra under the ordinary set-theoretic operations  $\cup, \cap$  and  $^c$ . Powerset Boolean algebras can be considered as the building blocks of all Boolean algebras. Stone's Representation Theorem asserts that every Boolean algebra is isomorphic to a subalgebra of some powerset Boolean algebra. In fact, every finite Boolean algebra is isomorphic to a powerset Boolean algebra.

**Example 0.1.12 :**

Another Boolean algebra which is of fundamental importance in the topological construction of Boolean powers is the subalgebra  $CL(X)$  of  $\mathcal{P}(X)$ , which consists of all the subsets of a topological space which are both open and closed. (Subsets of a topological space which are both open and closed are referred to as *clopen* subsets.) Stone had in fact shown that every Boolean algebra is isomorphic to  $CL(X)$  for some topological space  $X$ . This observation is significant for gaining insight into the relationship that exists between Boolean powers and certain algebras of continuous functions.

**Example 0.1.13 :**

A subset  $Y$  of a topological space  $X$  is regular open if it is equal to the interior of its closure: in symbols,

$$\overline{Y}^\circ = Y.$$

The set  $RO(X)$  which consists of all the regular open subsets of  $X$  can be given the structure of a Boolean algebra by defining the operations  $\wedge, \vee$  and  $^c$  on it as follows:

For  $Y, Z \in RO(X)$ ,

$$\begin{aligned} Y \vee Z &= \overline{Y \cup Z}^\circ, \\ Y \wedge Z &= Y \cap Z, \\ Y^c &= (X - Y)^\circ. \end{aligned}$$

The join ( $\vee$ ) and meet ( $\wedge$ ) operations on a Boolean algebra are finitary operations, and hence the join and meet of any finite subset of a Boolean algebra exist. However, the join and meet of an infinite subset of a Boolean algebra may not exist. It is noteworthy that, by defining  $\vee \mathcal{F} = \overline{\bigcup \mathcal{F}}^\circ$  and  $\wedge \mathcal{F} = \overline{\bigcap \mathcal{F}}^\circ$  for any family  $\mathcal{F} \subseteq RO(X)$ , the joins and meets of arbitrary subsets exist in  $RO(X)$ .

The following laws hold for the joins and meets of arbitrary subsets of a Boolean algebra.

**Lemma 0.1.14 :**

Suppose that  $B$  is a Boolean algebra and for  $C, D \subseteq B$ ,  $\vee C, \vee D \in B$ . Then

- (a)  $\vee C \vee \vee D = \vee\{c \vee d : c \in C, d \in D\}$ , (associativity)
- (b)  $\vee C \wedge \vee D = \vee\{c \wedge d : c \in C, d \in D\}$ , (distributivity)
- (c) for  $b \in B$ ,  $b \wedge \vee C = \vee\{b \wedge c : c \in C\}$ , (distributivity)
- (d)  $(\vee D)^c = \wedge\{d^c : d \in D\}$ . (de Morgan's law)

**Definition 0.1.15 :**

A Boolean algebra  $B$  is *complete* if for each  $C \subseteq B$ ,  $\vee C \in B$ .  $B$  is  $\kappa$ -*complete* if for each  $D \subseteq B$  such that  $|D| < \kappa$ ,  $\vee D \in B$ .

As mentioned in the introduction, the bounded Boolean power  $M[B]_\omega$  can be defined for any Boolean algebra  $B$ , but the Boolean power  $M[B]$  (the  $\kappa$ -bounded Boolean power  $M[B]_\kappa$ ) can only be defined if  $B$  is complete ( $\kappa$ -complete).

When working with a subalgebra  $A$  of a Boolean algebra  $B$ , it is essential to distinguish between  $\vee C$  (w.r.t.  $A$ ) and  $\vee C$  (w.r.t.  $B$ ) for  $C \subseteq A$ , since  $\vee C$  (w.r.t.  $A$ ) and  $\vee C$  (w.r.t.  $B$ ) may not coincide - in fact, the existence of  $\vee C$  (w.r.t.  $A$ ) does not guarantee the existence of  $\vee C$  (w.r.t.  $B$ ).

**Definition 0.1.16 :**

A subalgebra  $A$  of a Boolean algebra  $B$  is a *regular subalgebra* of  $B$  if for each  $C \subseteq A$  if  $\vee C$  (w.r.t.  $A$ ) exists then so does  $\vee C$  (w.r.t.  $B$ ) and  $\vee C$  (w.r.t.  $A$ ) =  $\vee C$  (w.r.t.  $B$ ).

In Chapter 5 it will be shown how an updirected set of regular subalgebras of  $B$  is used to define the notion of a limit Boolean ultrapower.

The following notion is significant for gaining insight into complete Boolean algebras.

**Definition 0.1.17 :**

A subset  $A$  of a partially ordered set  $B$  is *dense* in  $B$  if for each  $b \in B$  there exists  $a \in A$  such that  $a \leq b$ . In particular, if  $B$  is a Boolean algebra, then a subalgebra  $A$  of  $B$  is *dense* in  $B$  if for each  $b \in B - \{0\}$  there exists  $a \in A$  such that  $0 < a \leq b$ . The notion *dense* is equivalent to the notion *dually cofinal*.

**Proposition 0.1.18 :**

*Every dense subalgebra  $A$  of a Boolean algebra  $B$  is regular in  $B$ .*

The above proposition partly motivates the following definition.

**Definition 0.1.19 :**

A *completion* of a Boolean algebra  $B$  is a complete Boolean algebra  $\overline{B}$  such that  $B$  is a dense subalgebra of  $\overline{B}$ .

Now two complete Boolean algebras are isomorphic if they have dense subsets which are order-isomorphic with respect to the partial ordering induced by the Boolean operations. This implies that the completion of a Boolean algebra is unique, since if  $\overline{B}$  and  $\tilde{B}$  are completions of a Boolean algebra  $B$ , then  $B$  is dense in  $\overline{B}$  and in  $\tilde{B}$ .

By example 0.1.13,  $RO(X)$  is complete. Also, for any Boolean algebra  $B$ ,  $RO(B - \{0\})$  is a completion of  $B$ , and hence  $B \cong RO(B - \{0\})$  if  $B$  is complete.

The above statements are summed up in the following proposition.

**Proposition 0.1.20 :**

*A Boolean algebra  $B$  is complete iff it is isomorphic to  $RO(X)$  for some topological space  $X$ .*

Although a homomorphism between two Boolean algebras preserves the joins and meets of finite subsets, a stronger notion of a homomorphism is needed for the class of complete ( $\kappa$ -complete) Boolean algebras.

**Definition 0.1.21 :**

Suppose that  $A$  and  $B$  are Boolean algebras. A homomorphism  $f : A \rightarrow B$  is a *complete* ( $\kappa$ -complete) *homomorphism* if for each  $C \subseteq A$  (for each  $C \subseteq A$ ,  $|C| < \kappa$ ) such that  $\vee C \in A$ ,

$$f(\vee C) = \vee \{f(c) : c \in C\}.$$

**Definition 0.1.22 :**

A *filter* on a Boolean algebra  $B$  is a proper subset  $F$  of  $B$  such that

- (a) for all  $b_1, b_2 \in F$ ,  $b_1 \wedge b_2 \in F$ ,
- (b) for all  $b_1 \in F$  and  $b_2 \in B$ , if  $b_1 \leq b_2$ , then  $b_2 \in F$ .

Filters on a Boolean algebra can be partially ordered by set inclusion. A filter which is maximal with respect to this ordering is called an *ultrafilter*. Ultrafilters can also be characterised by the following proposition.

**Proposition 0.1.23 :**

*Suppose that  $F$  is a filter on a Boolean algebra  $B$ . Then  $F$  is an ultrafilter iff for each  $b \in B$  either  $b \in F$  or  $b^c \in F$ , but not both. (If  $b, b^c \in F$ , then  $0 = b \wedge b^c \in F$  which implies that  $F$  is not proper.)*

The following property identifies those subsets of a Boolean algebra which can be extended to filters, and hence by the application of Zorn's Lemma, to ultrafilters.

**Definition 0.1.24 :**

A subset  $C$  of a Boolean algebra  $B$  has the *finite intersection property* if for each  $c_1, \dots, c_n \in C$ ,

$$c_1 \wedge \dots \wedge c_n \neq 0.$$

**Lemma 0.1.25 :**

*Every subset of a Boolean algebra which has the finite intersection property can be extended to an ultrafilter. In particular, every non-zero element is contained in some ultrafilter, since the 1-element subset which contains this element has the finite intersection property and can hence be extended to an ultrafilter.*

The following definition describes one of the corner-stones in the construction of the Boolean power  $M[B]$ , namely a *partition*, since  $M[B]$  essentially constitutes functions defined either on a partition of  $B$ , or into a partition of  $B$ , as will be seen later.

**Definition 0.1.26 :**

A *partition* of a Boolean algebra  $B$  is a subset  $P$  of  $B$  such that

- (a) for all  $p_1, p_2 \in P$ , if  $p_1 \neq p_2$ , then  $p_1 \wedge p_2 = 0$ ,
- (b)  $\vee P = 1$ .

The following definition describes the construction of new Boolean algebras, namely the relative algebras, from ones that already exist. Relative algebras are important since certain properties of Boolean algebras are more easily discernible using appropriate relative algebras than the Boolean algebra itself. In this thesis, relative algebras will be used to construct the  $(B, P)$ -ultraproduct of models.

**Definition 0.1.27 :**

The *relative algebra* of the Boolean algebra  $B$  w.r.t.  $b \in B$  is the Boolean algebra

$$B \upharpoonright b = \{x \in B : x \leq b\}$$

in which the partial ordering is inherited from  $B$ .

**Remark 0.1.28 :**

$B \upharpoonright b$  is not a subalgebra of  $B$  if  $b \neq 1$ , since its greatest element and the complements of elements do not correspond with those in  $B$ . However,  $B \upharpoonright b$  is a homomorphic image of  $B$  under the homomorphism  $g^b : B \rightarrow B \upharpoonright b$  defined by

$$g^b(x) = x \wedge b.$$

**Lemma 0.1.29 :**

Suppose that  $B$  is a complete Boolean algebra and that  $P$  is a partition of  $B$ . Then

$$B \cong \Pi\{B \upharpoonright p : p \in P\}.$$

**Proof :**

Define functions  $g : B \rightarrow \Pi\{B \upharpoonright p : p \in P\}$  and  $h : \Pi\{B \upharpoonright p : p \in P\} \rightarrow B$  by

$$g(x) = \langle x \wedge p : p \in P \rangle,$$

$$h(\langle x_p : p \in P \rangle) = \vee\{x_p : p \in P\}.$$

Then  $g$  is a homomorphism since  $g(x) = \langle g^p(x) : p \in P \rangle$ , and  $g$  is 1-1 and onto since  $g$  and  $h$  are inverses.  $\square$

**0.2 The various constructions of  $M[B]$  and  $M[B]_\omega$** 

There are a number of different constructions of the Boolean power  $M[B]$  and the bounded Boolean power  $M[B]_\omega$ . At times, one construction may be more appropriate than any other in yielding new results or fulfilling any other objective of an author. For example, according to Banaschewski and Nelson [1980], the topological construction of Boolean powers is the most natural one, and by using it they show that  $M[B]$  preserves elementary equivalence in both arguments, that  $M[B]_\omega \equiv M[B]$ , and that the canonical embedding of  $M[B]_\omega$  into  $M[B]$  is elementary.

In this section, the various constructions of  $M[B]$  and  $M[B]_\omega$  will be presented in detail. The resulting Boolean powers will all be shown to be isomorphic.

### 0.2.1 A.L. Foster [1953]

Suppose that  $M$  is an algebra and  $B$  a complete Boolean algebra. The universe of the *Boolean power*  $M[B]$  is defined to be the set of all functions  $f : M \rightarrow B$  such that

- (a) for all  $m_1, m_2 \in M$ , if  $m_1 \neq m_2$ , then  $f(m_1) \wedge f(m_2) = 0$ ,
- (b)  $\vee \{f(m) : m \in M\} = 1$ .

Suppose that  $f_1, \dots, f_n \in M[B]$ . Each fundamental operation  $H$  on  $M$  can be extended to an operation on  $M[B]$  by defining

$$H(f_1, \dots, f_n)(m) = \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : H(m_1, \dots, m_n) = m\}.$$

The *bounded Boolean power*  $M[B]_\omega$  is obtained by adding a third requirement to the definition of  $M[B]$ , namely

- (c)  $|\{m \in M : f(m) \neq 0\}| < \aleph_0$ .

Thus if  $M$  is finite, or  $B$  is finite, then  $M[B] = M[B]_\omega$ .

#### Remark 0.2.1.1 :

Although Foster has defined the Boolean power  $M[B]$  in the case that  $M$  is an algebra, the definition is nowadays used in a more general sense where  $M$  is any model. The construction of Foster is the one most commonly found in the literature, and it will be the one most frequently used in this thesis.

### 0.2.2 B. Banaschewski and E. Nelson [1980]

Although the topological construction of  $M[B]$  and  $M[B]_\omega$  as given by Banaschewski and Nelson [1980] will not be referred to again in this thesis, it is included for the sake of completeness.

Before proceeding with a discussion on the construction of Banaschewski and Nelson, a few basic topological definitions and results which can be found in Bell and Slomson [1971] are required to put the discussion in context.

#### Definition 0.2.2.1 :

A *Boolean space* is a compact Hausdorff space with a base of clopen sets. The elements of this base, being closed subsets of a compact space, are themselves compact.

**Definition 0.2.2.2 :**

A subspace  $Y$  of a topological space  $X$  is *connected* if there exists no non-empty open sets  $C$  and  $D$ , with  $C \cap D = \emptyset$ , such that  $Y = C \cup D$ .  $Y$  is a *component* of  $X$  if  $Y$  is a maximal connected subspace of  $X$ .  $X$  is *totally disconnected* if each component of  $X$  has only one element.

**Lemma 0.2.2.3 :**

*A Boolean space is totally disconnected.*

**Proof :**

Suppose that  $X$  is a Boolean space and  $Y$  is a subspace of  $X$  such that  $y_1, y_2 \in Y$ ,  $y_1 \neq y_2$ . Since  $X$  is a Hausdorff space, and since  $X$  has a base which consists of clopen sets, there exists a clopen set  $U$  such that  $y_1 \in U$ , but  $y_2 \notin U$ . Then

$$Y = (Y \cap U) \cup (Y - U),$$

and hence  $Y$  is not connected. □.

Recall from Example 0.1.12 that  $CL(X)$  is the subalgebra of  $\mathcal{P}(X)$  which consists of the clopen subsets of the topological space  $X$ . If  $X$  is a Boolean space, then  $CL(X)$  forms a base for the topology of  $X$ . The following result shows that  $CL(X)$  is uniquely characterised in this way.

**Lemma 0.2.2.4 :**

*Suppose that  $X$  is a Boolean space and  $\mathcal{A}$  is a subalgebra of  $\mathcal{P}(X)$  which forms a base for the topology of  $X$ . Then  $\mathcal{A} = CL(X)$ .*

**Proof :**

Suppose that  $U \in \mathcal{A}$ . Then  $U$  is open since  $\mathcal{A}$  forms a base for the topology of  $X$ . Since  $\mathcal{A}$  is a subalgebra of  $\mathcal{P}(X)$ , it is closed under the formation of complements, and hence  $U$  is also closed. Thus  $U \in CL(X)$ .

Conversely, suppose that  $U \in CL(X)$ . Since  $U$  is open and  $\mathcal{A}$  is a base for the topology of  $X$ , for each  $u \in U$  there exists  $V_u \in \mathcal{A}$  such that  $u \in V_u \subseteq U$ .

$\{V_u : u \in U\}$  forms an open cover for  $U$ . Since  $U$  is also closed, and  $X$  is compact,  $U$  is compact and hence  $\{V_u : u \in U\}$  has a finite subcover  $\{V_{u_i} : 1 \leq i \leq n\}$ . Thus

$$U = V_{u_1} \cup \cdots \cup V_{u_n}.$$

Since  $\mathcal{A}$  is a subalgebra, it is closed under finite unions. Hence  $U \in \mathcal{A}$ . □

**Definition 0.2.2.5 :**

The *Stone space*  $ULT(B)$  of the Boolean algebra  $B$  is the set of all ultrafilters of  $B$  which has been given the topology which has as a base the set  $\{\mathcal{F}_b : b \in B\}$  of subsets of  $ULT(B)$ , where

$$\mathcal{F}_b = \{F \in ULT(B) : b \in F\}.$$

**Theorem 0.2.2.6 [M.H. Stone 1936]:**

*Suppose that  $B$  is a Boolean algebra. Then  $ULT(B)$  is a Boolean space and  $CL(ULT(B)) \cong B$ .*

**Proof :**

$$\{\mathcal{F}_b : b \in B\} \cong B :$$

Suppose that  $b_1, b_2 \in B$  and  $b_1 \neq b_2$ . Now either  $b_1 \not\leq b_2$  or  $b_2 \not\leq b_1$ , say  $b_1 \not\leq b_2$ . Then  $b_1 \wedge b_2 \neq b_1$ , and thus  $b_1 \wedge b_2^c \neq (b_1 \wedge b_2) \wedge b_2^c = b_1 \wedge (b_2 \wedge b_2^c) = b_1 \wedge 0 = 0$ . Hence  $\{b_1, b_2^c\}$  has the finite intersection property and can thus be extended to an ultrafilter  $F$ . Since  $b_2^c \in F, b_2 \notin F$ , and thus  $\mathcal{F}_{b_1} \neq \mathcal{F}_{b_2}$ .

Now for each  $b_1, b_2 \in B$  and each filter  $F$  of  $B$ ,  $b_1 \wedge b_2 \in F$  iff  $b_1 \in F$  and  $b_2 \in F$ . Hence it follows that  $\mathcal{F}_{b_1} \cap \mathcal{F}_{b_2} = \mathcal{F}_{b_1 \wedge b_2}$ . Also for each  $F, b \in F$  iff  $b^c \notin F$  which implies that  $\mathcal{F}_{b^c} = ULT(B) - \mathcal{F}_b$ .

$ULT(B)$  is Hausdorff:

Suppose that  $F, G \in ULT(B)$  and  $F \neq G$ . Then there exists  $b \in B$  such that  $b \in F$ , but  $b \notin G$  which implies that  $b^c \in G$ . Thus  $\mathcal{F}_b$  and  $\mathcal{F}_{b^c}$  are open subsets of  $ULT(B)$  such that  $F \in \mathcal{F}_b, G \in \mathcal{F}_{b^c}$  and  $\mathcal{F}_b \cap \mathcal{F}_{b^c} = \emptyset$ .

$ULT(B)$  is compact :

Suppose that  $\{\mathcal{F}_{b_i} : i \in I\}$  is a cover of  $ULT(B)$  which has no finite subcover. Then for each finite subset  $J$  of  $I$ ,

$$\cup\{\mathcal{F}_{b_i} : i \in J\} \neq ULT(B),$$

and hence

$$\mathcal{F}_0 = \emptyset \neq \cap\{\mathcal{F}_{b_i^c} : i \in J\} = \mathcal{F}_{\bigwedge\{b_i^c : i \in J\}}$$

Thus

$$\bigwedge\{b_i^c : i \in J\} \neq 0,$$



and so  $\{b_i^c : i \in I\}$  has the finite intersection property and can therefore be extended to an ultrafilter  $F$  of  $B$ . For each  $i \in I$ ,  $b_i^c \in F$ , and hence

$$F \notin \cup\{\mathcal{F}_{b_i} : i \in I\},$$

which contradicts the assumption that  $\{\mathcal{F}_{b_i} : i \in I\}$  covers  $ULT(B)$ .

$ULT(B)$  has a base of clopen sets:

Since  $\{\mathcal{F}_b : b \in B\}$  is a base for  $ULT(B)$ ,  $\mathcal{F}_b$  is open for each  $b \in B$ , and since  $\mathcal{F}_b = ULT(B) - \mathcal{F}_{b^c}$ ,  $\mathcal{F}_b$  is also closed.

By Lemma 0.2.2.4,  $CL(ULT(B)) = \{\mathcal{F}_b : b \in B\}$ , and since  $\{\mathcal{F}_b : b \in B\} \cong B$ , the result follows.  $\square$

The following result is the topological dual to Theorem 0.2.2.6 which was originally due to M.H. Stone. Together with Theorem 0.2.2.6, it shows that, in a certain sense, the operations  $ULT$  and  $CL$  are “inverses”, and that every Boolean algebra can be uniquely associated with a Boolean space, and vice versa. Expressed in categorical terms, the contravariant functors  $CL : BS \rightarrow BA$  and  $ULT : BA \rightarrow BS$ , where  $BA$  is the category of Boolean algebras and their homomorphisms and  $BS$  the category of Boolean spaces and continuous functions, are adjoint to each other.

**Theorem 0.2.2.7 :**

*Suppose that  $X$  is a Boolean space. Then  $X$  is homeomorphic to  $ULT(CL(X))$ .*

**Proof :**

Define  $\mathcal{U}_x$  by

$$\mathcal{U}_x = \{U \in CL(X) : x \in U\}.$$

$\mathcal{U}_x$  is a filter on  $CL(X)$ , and since for each  $U \in CL(X)$ , either  $x \in U$  or  $x \in CL(X) - U$ ,  $\mathcal{U}_x \in ULT(CL(X))$ . Suppose that  $x_1, x_2 \in X$  and  $x_1 \neq x_2$ . Since  $X$  is Hausdorff, there exist  $U_1, U_2 \in CL(X)$  such that  $x_1 \in U_1, x_2 \in U_2$  and  $U_1 \cap U_2 = \emptyset$ . Thus  $\mathcal{U}_{x_1} \neq \mathcal{U}_{x_2}$ . Now suppose that  $\mathcal{V}$  is an ultrafilter on  $CL(X)$ . Since  $X$  is compact and  $\mathcal{V}$  is a family of closed subsets with the finite intersection property, there exists  $x \in X$  such that  $x \in V$  for each  $V \in \mathcal{V}$ . Then  $\mathcal{V} \subseteq \mathcal{U}_x$ , and since  $\mathcal{V}$  is an ultrafilter,  $\mathcal{V} = \mathcal{U}_x$ , which shows that the function  $x \rightarrow \mathcal{U}_x$  is onto. Hence for each  $U \in CL(X)$ ,  $\{\mathcal{V} \in ULT(CL(X)) : U \in \mathcal{V}\} = \{\mathcal{U}_x : x \in U\}$ , so that  $x \rightarrow \mathcal{U}_x$  maps the base for the topology of  $X$  onto the base  $CL(ULT(CL(X)))$  for the topology of  $ULT(CL(X))$ . Hence  $x \rightarrow \mathcal{U}_x$  is a homeomorphism.  $\square$

Now that the basic topological definitions and results required have been presented, the discussion will focus on the actual construction of Banaschewski and Nelson.

Suppose that  $X$  is a topological space and  $M$  is an algebra which has been given the discrete topology. The algebra  $C(X, M)$  is the algebra of continuous functions  $f : X \rightarrow M$  with operations defined pointwise. It should be noted that  $C(X, M)$  is isomorphic to  $C(Y, M)$  where  $Y$  is the component space of  $X$ . Also, if  $X$  is discrete, then every function  $f : X \rightarrow M$  is continuous. In the event that  $M$  is the 2-element Boolean algebra,  $C(X, M)$  is isomorphic to  $CL(X)$  for each topological space  $X$ , and thus, by applying the Stone duality,  $C(ULT(B), 2)$  is isomorphic to  $B$  for all Boolean algebras  $B$ .

Now suppose that  $X$  is compact. For each  $f \in C(X, M)$ ,  $f[X]$  is a compact subset of a discrete space and is thus finite. Hence each  $f \in C(X, M)$  determines a function  $\hat{f} : M \rightarrow CL(X)$  defined by

$$\hat{f}(m) = f^{-1}(\{m\})$$

such that

- (a)  $\hat{f}(m_1) \cap \hat{f}(m_2) = \emptyset$  if  $m_1 \neq m_2$ ,
- (b)  $\cup\{\hat{f}(m) : m \in M\} = X$ ,
- (c)  $|\{m : \hat{f}(m) \neq \emptyset\}| < \aleph_0$ .

Furthermore, suppose that  $H$  is an operation on  $M$  and  $f_1, \dots, f_n \in C(X, M)$ . Then for each  $m \in M$

$$\begin{aligned} H(\widehat{f_1, \dots, f_n})(m) &= H(f_1, \dots, f_n)^{-1}(\{m\}) \\ &= \{x \in X : H(f_1, \dots, f_n)(x) = m\} \\ &= \{x \in X : H(f_1(x), \dots, f_n(x)) = m\} \\ &= \cup\{\hat{f}_1(m_1) \cap \dots \cap \hat{f}_n(m_n) : H(m_1, \dots, m_n) = m\}. \end{aligned}$$

Now if  $X$  is partitioned into clopen sets and  $f : X \rightarrow M$  is constant on each of these, then  $f \in C(X, M)$ . Hence for any compact topological space  $X$ ,  $C(X, M)$  is isomorphic to the bounded Boolean power  $M[CL(X)]_\omega$ . In particular,  $C(ULT(B), M)$  is isomorphic to  $M[CL(ULT(B))]_\omega$ , and by the Stone duality it follows that  $C(ULT(B), M)$  is isomorphic to  $M[B]_\omega$ .

For the construction of the Boolean power, suppose that  $D(X, M)$  is the algebra of all functions  $f : X \rightarrow M$  which are continuous on some dense open subset of  $X$  with operations defined pointwise.  $D(X, M)$  is a subalgebra of the algebra

of all functions  $f : X \rightarrow M$  since all the operations are finitary, and finite intersections of dense open sets are dense open. Define a congruence relation  $\sim$  on  $D(X, M)$  by

$$f \sim g \text{ iff there exists some dense open set } U \text{ such that } f \upharpoonright U = g \upharpoonright U.$$

For each function  $f : X \rightarrow M$ , let  $\text{dom}_c f$  be the subset of  $X$  on which  $f$  is continuous. Then  $\text{dom}_c f$  is open, and  $f \in D(X, M)$  iff  $\text{dom}_c f$  is dense. The congruence  $\sim$  can also be given by

$$f \sim g \text{ iff } f \upharpoonright \text{dom}_c f \cap \text{dom}_c g = g \upharpoonright \text{dom}_c f \cap \text{dom}_c g :$$

Since  $\text{dom}_c f \cap \text{dom}_c g$  is dense open, the sufficiency is clear. Suppose then that  $f \upharpoonright U = g \upharpoonright U$  for some dense open  $U$ . Then  $f \upharpoonright \text{dom}_c f \cap \text{dom}_c g \cap U = g \upharpoonright \text{dom}_c f \cap \text{dom}_c g \cap U$ , and since  $\text{dom}_c f \cap \text{dom}_c g \cap U$  is dense in  $\text{dom}_c f \cap \text{dom}_c g$ ,  $f \upharpoonright \text{dom}_c f \cap \text{dom}_c g = g \upharpoonright \text{dom}_c f \cap \text{dom}_c g$ .

Each  $f \in D(X, M)$  determines a function  $\hat{f} : M \rightarrow RO(X)$  defined by

$$\hat{f}(m) = \overline{f^{-1}(\{m\}) \cap \text{dom}_c f}^\circ$$

such that

- (a)  $\hat{f}(m_1) \cap \hat{f}(m_2) = \emptyset$  if  $m_1 \neq m_2$ ,
- (b)  $\bigvee \{\hat{f}(m) : m \in M\} = X$ .

Also, for  $f, g \in D(X, M)$ ,  $f \sim g$  iff  $\hat{f} = \hat{g}$  :

Suppose that  $f \sim g$ . Then  $f \upharpoonright \text{dom}_c f \cap \text{dom}_c g = g \upharpoonright \text{dom}_c f \cap \text{dom}_c g$  and thus for each  $m \in M$ ,

$$\begin{aligned} \hat{f}(m) &= \overline{f^{-1}(\{m\}) \cap \text{dom}_c f}^\circ \\ &= \overline{f^{-1}(\{m\}) \cap \text{dom}_c f \cap \text{dom}_c g}^\circ \\ &= \overline{g^{-1}(\{m\}) \cap \text{dom}_c f \cap \text{dom}_c g}^\circ \\ &= \overline{g^{-1}(\{m\}) \cap \text{dom}_c g}^\circ \\ &= \hat{g}(m). \end{aligned}$$

Conversely, suppose that  $\hat{f} = \hat{g}$ . Then

$$f^{-1}(\{m\}) \cap \text{dom}_c f \cap \text{dom}_c g = g^{-1}(\{m\}) \cap \text{dom}_c f \cap \text{dom}_c g$$

for each  $m \in M$ . Hence  $f \upharpoonright \text{dom}_c f \cap \text{dom}_c g = g \upharpoonright \text{dom}_c f \cap \text{dom}_c g$ .

Furthermore, suppose that  $H$  is an operation on  $M$ ,  $f_1, \dots, f_n \in D(X, M)$  and  $V = \text{dom}_c f_1 \cap \dots \cap \text{dom}_c f_n$ .

Then for each  $m \in M$ ,

$$\begin{aligned}
& H(\widehat{f_1, \dots, f_n})(m) \\
&= \overline{H(f_1, \dots, f_n)^{-1}(\{m\}) \cap V}^\circ \\
&= \overline{\{x \in V : H(f_1(x), \dots, f_n(x)) = m\}}^\circ \\
&= \overline{\cup\{f_1^{-1}(\{m_1\}) \cap \dots \cap f_n^{-1}(\{m_n\}) \cap V : H(m_1, \dots, m_n) = m\}}^\circ \\
&= \overline{(\cup\{f_1^{-1}(\{m_1\}) \cap \dots \cap f_n^{-1}(\{m_n\}) \cap V : H(m_1, \dots, m_n) = m\})}^\circ \\
&= \overline{(\cup\{f_1^{-1}(\{m_1\}) \cap V \cap \dots \cap f_n^{-1}(\{m_n\}) \cap V : H(m_1, \dots, m_n) = m\})}^\circ \\
&= \vee\{\hat{f}_1(m_1) \wedge \dots \wedge \hat{f}_n(m_n) : H(m_1, \dots, m_n) = m\}.
\end{aligned}$$

Now for each function  $h : M \rightarrow RO(X)$ ,  $\vee\{h(m) : m \in M\} = X$  iff  $\cup\{h(m) : m \in M\}$  is dense. Hence for any topological space  $X$ ,  $D(X, M)/\sim$  is isomorphic to the Boolean power  $M[RO(X)]$ . In particular,  $D(ULT(B), M)/\sim$  is isomorphic to  $M[RO(ULT(B))]$ , and since for any Boolean space  $X$ ,  $RO(X)$  is the completion of  $CL(X)$ , it follows that for a complete Boolean algebra  $B$ ,  $D(ULT(B), M)/\sim$  is isomorphic to  $M[B]$ . It is noteworthy that if  $B$  is not complete and  $M$  is the 2-element Boolean algebra, then  $D(ULT(B), M)/\sim$  is the completion of  $B : RO(ULT(B))$  is isomorphic to  $C(ULT(RO(ULT(B))), 2)$  which is in turn isomorphic to  $D(ULT(RO(ULT(B))), 2)/\sim$  and the result follows by the Stone duality.

### 0.2.3 M. Benda [1974]

Suppose that  $M$  is a model and  $B$  is a complete Boolean algebra. The universe of the Boolean power  $M[B]$  is the set of all functions  $f : B \rightarrow M$  such that the domain of  $f$  is a partition of  $B$ .

Suppose that  $f_1, \dots, f_n \in M[B]$  with domains  $P_1, \dots, P_n$  respectively. Each relation  $R$  on  $M$  can be extended to a relation on  $M[B]$  by defining

$$R(f_1, \dots, f_n) = \vee\{p_1 \wedge \dots \wedge p_n : p_i \in P_i \text{ for } 1 \leq i \leq n \text{ and } M \models R(f_1(p_1), \dots, f_n(p_n))\}.$$

Now for each  $f \in M[B]$  define a function  $I(f)$  by

$$I(f)(m) = \begin{cases} f^{-1}(m), & m \in \text{range } f, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $I(f) \in M[B]$  as originally defined by Foster [1953] and  $I$  is an isomorphism between the Boolean power described above and that as defined by Foster.

#### 0.2.4 P. Ouwehand and H. Rose [1998]

Partitions and direct limits are the corner-stones of the construction of Ouwehand and Rose and hence the discussion is introduced by further remarks and some preliminary results on partitions, filters and direct limits, the last notion being motivated by the observation that a  $\kappa$ -complete Boolean algebra may be regarded as the direct limit of an updirected set of powerset Boolean algebras.

Suppose that  $B$  is a Boolean algebra and  $\mathbb{P}_B$  is the set of all partitions of  $B$ . For  $P, Q \in \mathbb{P}_B$ ,  $P$  is said to refine  $Q$  (or  $P$  is a refinement of  $Q$ , or  $Q$  is coarser than  $P$ ) iff for each  $p \in P$  there exists some  $q \in Q$  such that  $p \leq q$ . A partial ordering can be imposed on  $\mathbb{P}_B$  by writing  $P \leq Q$  iff  $Q$  is a refinement of  $P$ . Note that the partial ordering is reverse to the refinement relation between partitions. Now if  $P \leq Q$ , and  $p \in P$ , then

$$p = \vee \{q \in Q : q \leq p\}.$$

Also, if  $P, Q \in \mathbb{P}_B$ , then

$$T = \{p \wedge q : p \in P, q \in Q \text{ and } p \wedge q \neq 0\}$$

is the coarsest common refinement of  $P$  and  $Q$ . It follows that  $\mathbb{P}_B$  is an upper semilattice with respect to this ordering, and is therefore an updirected set. The following subsemilattices of  $\mathbb{P}_B$  are distinguished:

- (a)  $\mathbb{P}_B(P)$ , the principal filter generated by  $P \in \mathbb{P}_B$ ,
- (b)  $\mathbb{P}_B^\kappa = \{P \in \mathbb{P}_B : |P| < \kappa\}$ , for some infinite cardinal  $\kappa$ .

If  $B$  is  $\kappa$ -complete, then for each  $P \in \mathbb{P}_B^\kappa$ , a complete Boolean algebra embedding  $i_P : \mathcal{P}(P) \rightarrow B$  can be defined by

$$i_P(U) = \vee U.$$

It is clear that  $i_P$  has the asserted properties. Furthermore, if  $P \leq Q$ , then there exists a function  $i_{PQ} : \mathcal{P}(P) \rightarrow \mathcal{P}(Q)$  defined by

$$i_{PQ}(U) = \{q \in Q : \exists p \in P [q \leq p]\}.$$

The following can easily be shown:

(a) If  $P \leq Q \leq T$ , then

$$i_{QT} \circ i_{PQ} = i_{PT}.$$

(a) If  $P \leq Q$ , then

$$i_P = i_Q \circ i_{PQ}.$$

**Remark 0.2.4.1 :**

If  $B$  is  $\kappa$ -complete and  $P \in \mathbb{P}_B^\kappa$ , then in order to simplify the notation, the complete subalgebra of  $B$  under the embedding  $i_P$  will be identified with the powerset Boolean algebra  $\mathcal{P}(P)$ . Since  $P \leq Q$  implies that  $\mathcal{P}(P) \subseteq \mathcal{P}(Q)$ , the family

$$\{\mathcal{P}(P) : P \in \mathbb{P}_B^\kappa\}$$

ordered by inclusion is order-isomorphic to  $\mathbb{P}_B^\kappa$ , and is therefore also an updirected set.

The following result can now be proved.

**Lemma 0.2.4.2 [P. Ouwehand and H. Rose 1998]:**

*Suppose that  $B$  is a Boolean algebra.*

- (a)  $B = \cup\{\mathcal{P}(P) : P \in \mathbb{P}_B^{\aleph_0}\}.$
- (b) *If  $B$  is  $\kappa$ -complete, then for  $\aleph_0 \leq \eta \leq \kappa$ ,*

$$B = \cup\{\mathcal{P}(P) : P \in \mathbb{P}_B^\eta\}.$$

- (c) *If  $B$  is complete, then for each  $P \in \mathbb{P}_B$ ,*

$$B = \cup\{\mathcal{P}(Q) : Q \in \mathbb{P}_B(P)\}.$$

**Definition 0.2.4.3 :**

Suppose that  $\mathcal{U}$  is an updirected set and  $\{B_P : P \in \mathcal{U}\}$  a family of complete Boolean algebras with complete embeddings

$$i_{PQ} : B_P \rightarrow B_Q, \quad P \leq Q,$$

such that

$$i_{QT} \circ i_{PQ} = i_{PT}, \quad P \leq Q \leq T.$$

For  $b_P \in B_P, b_Q \in B_Q$ , define an equivalence relation  $\sim$  on  $\cup\{B_P : P \in \mathcal{U}\}$  by

$$b_P \sim b_Q \text{ iff there exists } T \geq P, Q \text{ such that } i_{PT}(b_P) = i_{QT}(b_Q).$$

Then the *direct limit*  $B$  of  $\{B_P : P \in \mathcal{U}\}$  is the set of all equivalence classes of  $\cup\{B_P : P \in \mathcal{U}\}$  with respect to  $\sim$ , with the meets, joins and complements defined in the obvious way.

**Remark 0.2.4.4 :**

- (a) By Lemma 0.2.4.2, a  $\kappa$ -complete Boolean algebra is the direct limit of powerset Boolean algebras.
- (b) The embeddings  $i_P, i_{PQ}$  will be regarded as inclusion functions. Thus for  $P \leq Q$ ,  $B_P$  is a complete subalgebra of  $B_Q$ , and for each  $P \in \mathcal{U}$ ,  $B_P$  is a complete subalgebra of  $B$ .
- (c) In general, the direct limit of complete Boolean algebras is not complete. For example, if  $B_n = \mathcal{P}(n)$  for  $n < \aleph_0$ , then the direct limit is the finite-cofinite Boolean algebra which is not complete.

The following condition ensures that the direct limit is complete and will be used to show that every  $\kappa$ -bounded Boolean ultrapower is the direct limit of ordinary ultrapowers.

**Definition 0.2.4.5 [P. Ouwehand and H. Rose 1998] :**

Suppose that  $B$  is the direct limit of  $\{B_P : P \in \mathcal{U}\}$ .  $B$  has the  $\kappa$ -*partition cofinality property* with respect to  $\{B_P : P \in \mathcal{U}\}$  if for each disjoint  $X \subseteq B$ ,  $|X| < \kappa$ , there exist  $Q \in \mathbb{P}_B^\kappa$ ,  $P \in \mathcal{U}$  and a partition  $T$  of  $B_P$  such that  $X \subseteq Q \leq T$ .

**Lemma 0.2.4.6 [P. Ouwehand and H. Rose 1998]:**

*Suppose that  $B$  is the direct limit of  $\{B_P : P \in \mathcal{U}\}$  and  $B$  has the  $\kappa$ -partition cofinality property with respect to  $\{B_P : P \in \mathcal{U}\}$ . Then  $B$  is  $\kappa$ -complete.*

**Proof :**

Suppose that  $X \subseteq B$  is disjoint and  $|X| < \kappa$ . It suffices to show that  $\vee X \in B$ . Since  $X$  is disjoint, there exists  $Q \in \mathbb{P}_B^\kappa$  such that  $X \subseteq Q$ . Choose  $P \in \mathcal{U}$  and a partition  $T$  of  $B_P$  such that  $Q \leq T$ . Suppose that  $Y = \{t \in T : \exists q \in X[t \leq q]\}$ . Then  $\vee Y \in B_P$  since  $B_P$  is complete, and since  $B_P$  is a subalgebra of  $B$ ,  $\vee Y \in B$ . The result follows from the observation that  $\vee X = \vee Y$ .  $\square$

A family of filters  $\{F_P : P \in \mathcal{U}\}$  is updirected iff  $F_P \subseteq F_Q$  whenever  $P \leq Q \in \mathcal{U}$  i.e.

$$F_P = F_Q \cap B_P, \quad P \leq Q.$$

The following can easily be observed.

**Lemma 0.2.4.7 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $B$  is the direct limit of  $\{B_P : P \in \mathcal{U}\}$ . Then*

- (a) *The union of an updirected family  $\{F_P : P \in \mathcal{U}\}$ , where for each  $P \in \mathcal{U}$ ,  $F_P$  is an (ultra-) filter on  $B_P$ , is an (ultra-) filter on  $B$ .*
- (b) *If  $F$  is an (ultra-) filter on  $B$ , then the family  $\{F_P : P \in \mathcal{U}\}$ , where for each  $P \in \mathcal{U}$ ,  $F_P = F \cap B_P$ , is an updirected family of (ultra-) filters.*

Now suppose that  $B$  is a complete Boolean algebra and  $F$  is a filter on  $B$ . Suppose that  $P \in \mathbb{P}_B$  such that  $F \cap P = \emptyset$ . For  $u \in F$ , define

$$X_u^P = \{p \in P : p \wedge u \neq 0\}.$$

Clearly

$$u = \vee \{p \wedge u : p \in P \text{ and } p \wedge u \neq 0\} \leq \vee X_u^P.$$

**Lemma 0.2.4.8 [P. Ouwehand and H. Rose 1998] :**

*$F_P = \{\vee X_u^P : u \in F\}$  is a filter on the complete subalgebra  $\mathcal{P}(P)$  of  $B$ . Furthermore, if  $F$  is an ultrafilter on  $B$ , then  $F_P$  is an ultrafilter on  $\mathcal{P}(P)$ .*

**Proof :**

Since the embedding  $i_P : \mathcal{P}(P) \rightarrow B$  is given by

$$i_P(X) = \vee X,$$

it suffices to show that  $\{X_u^P : u \in F\}$  is an (ultra-)filter on the powerset Boolean algebra  $\mathcal{P}(P)$ . For  $u \in F$  and  $X_u^P \subseteq Y \subseteq P$ ,  $u \leq \vee X_u^P \leq \vee Y$ . Hence  $v = \vee Y \in F$  and  $Y = X_v^P$ . Thus  $Y \in \{X_u^P : u \in F\}$ . Now suppose that  $u, v \in F$  and  $w = u \wedge v$ . Then

$$\begin{aligned} w = u \wedge v &= \vee \{p \wedge u : p \in X_u^P\} \wedge \vee \{\tilde{p} \wedge v : \tilde{p} \in X_v^P\} \\ &= \vee \{p \wedge \tilde{p} \wedge w : p \in X_u^P, \tilde{p} \in X_v^P\}. \end{aligned}$$

Since  $w \neq 0$ , and for  $p, \tilde{p} \in P$ ,  $p \wedge \tilde{p} = 0$  whenever  $p \neq \tilde{p}$ ,  $X_u^P \cap X_v^P \neq \emptyset$ . Finally, suppose that  $F$  is an ultrafilter on  $B$ , and  $U$  and  $V$  are disjoint subsets of  $P$  such that  $U \cup V = P$ . Then either  $\vee U \in F$  or  $\vee V \in F$ , and hence either  $U$  or  $V$  is an element of  $\{X_u^P : u \in F\}$ .  $\square$

The following is an immediate consequence of Lemma 0.2.4.8.



**Corollary 0.2.4.9 :**

Suppose that  $F$  and  $P$  are as in Lemma 0.2.4.8. Then

$$F = \cup \{F_Q : Q \in \mathbb{P}_B(P)\}.$$

**Remark 0.2.4.10 :**

The above corollary holds for any cofinal subset of  $\mathbb{P}_B$ .

Now that the preliminary results have been stated, the discussion will focus on the construction of a  $\kappa$ -bounded Boolean power.

Suppose that  $M$  is a model,  $I$  is a set and  $f_1, \dots, f_n \in M^I$ . Each relation  $R$  on  $M$  can be extended to a relation on the direct power  $M^I$  by defining

$$M \models R(f_1, \dots, f_n) \text{ iff } M \models R(f_1(i), \dots, f_n(i)) \text{ for each } i \in I.$$

Algebraic operations can be defined similarly. The direct power  $M^I$  is thus a model of the same type as  $M$ . In what follows, the  $\kappa$ -bounded Boolean power  $M[B]_\kappa$  will be constructed as a direct limit of such powers.

Suppose that  $B$  is a  $\kappa$ -complete Boolean algebra, and suppose that  $B^+ = \{b \in B : b \neq 0\}$ . Suppose that  $P \in \mathbb{P}_B^\kappa$  and  $f \in M^P$ . The domain of  $f$  can be regarded as being  $\{b \in B^+ : \exists p \in P[b \leq p]\}$  by defining

$$f(b) = f(p), \quad b \leq p, p \in P.$$

Thus if  $f$  is defined on  $P$ , it is also defined on any refinement  $Q$  of  $P$ , yielding functions

$$i_{PQ} : M^P \rightarrow M^Q$$

with the usual commutativity property

$$i_{PT} = i_{QT} \circ i_{PQ}, \quad P \leq Q \leq T.$$

Consider  $\cup \{M^P : P \in \mathbb{P}_B^\kappa\}$ , and define an equivalence relation  $\sim$  on  $\cup \{M^P : P \in \mathbb{P}_B^\kappa\}$  as follows : For  $f \in M^P, g \in M^Q$ ,

$$f \sim g \text{ iff } f \text{ and } g \text{ agree on some common refinement of } P \text{ and } Q.$$

Such a common refinement will be referred to as a *common domain* of  $f$  and  $g$ . Then

$$M[B]_\kappa = \cup\{M^P : P \in \mathbb{P}_B^\kappa\} / \sim,$$

with relations

$$M[B]_\kappa \models R(f_1 / \sim, \dots, f_n / \sim)$$

iff there exists a common domain  $P$  of  $f_1, \dots, f_n$  such that

$$M \models R(f_1(p), \dots, f_n(p)) \text{ for each } p \in P.$$

Note that the  $\kappa$ -completeness of  $B$  is assumed whenever the existence of the  $\kappa$ -bounded Boolean power  $M[B]_\kappa$  is assumed.

**Remark 0.2.4.11 :**

- (a) Each equivalence class  $f / \sim$  has a unique representative  $g$  such that  $g$  is injective on its domain. Hence  $f$  rather than  $f / \sim$  will be used to denote an element of  $M[B]_\kappa$ .
- (b) The model  $M$  can be identified with the submodel of  $M[B]_\kappa$  which consists of all those functions which have the partition  $\{1\}$  as their domain.
- (c) The powerset Boolean algebra  $\mathcal{P}(I)$  has a unique finest partition which consists of all the singletons. Thus

$$M[\mathcal{P}(I)] \cong M^I.$$

- (d) If  $B$  is complete, then it follows from Lemma 0.2.4.2(c) that for  $P \in \mathbb{P}_B$ ,  $M[B]$  is the direct limit of  $\{M^Q : Q \in \mathbb{P}_B(P)\}$ .
- (e) If  $B$  is  $\kappa$ -complete and  $\chi$  is cofinal in  $\mathbb{P}_B^\kappa$ , then by the nature of the direct limit,  $M[B]_\kappa$  is the direct limit of  $\{M^P : P \in \chi\}$ .

The discussion on the construction of Ouwehand and Rose [1998] is concluded with the following well-known result of Frayne, Morel and Scott [1962].

**Theorem 0.2.4.12 [T. Frayne, A. Morel, D. Scott 1962] :**

*Suppose that  $\mathcal{U}$  is an updirected set and  $M$  is the direct limit of  $\{M_P : P \in \mathcal{U}\}$ . Then  $M$  is embeddable into some ultraproduct  $\Pi\{M_P : P \in \mathcal{U}\} / F$ .*

**Corollary 0.2.4.13 [P. Ouwehand and H. Rose 1998] :**

*Every  $\kappa$ -bounded Boolean power of  $M$  is embeddable into an ultraproduct of direct powers of  $M$ .*

If  $B$  is a complete Boolean algebra (i.e.  $B$  is  $\kappa$ -complete, where  $\kappa = |B|^+$ ), then the Boolean power  $M[B]$  as constructed by Ouwehand and Rose [1998] is

isomorphic to the Boolean power  $M[B]$  as defined by Benda [1974]. (See Section 0.2.3.) Although the elements of  $M[B]$  of Ouwehand and Rose are equivalence classes, if  $f$  and  $g$  are equivalent, then the description of the equality relation implies that  $f = g$  in  $M[B]$  as defined by Benda.

For the sake of completeness, the two constructions of  $M[B]_\omega$  as given by Hodges [1993] are mentioned below, even though no further reference to them will be made in this thesis.

The first construction requires that  $B$  be a subalgebra of some powerset algebra  $\mathcal{P}(I)$  and describes  $M[B]_\omega$  as a submodel of the direct power  $M^I$ . This construction is in essence the same as that of Benda [1974], because by the Stone Representation Theorem, every Boolean algebra is isomorphic to a subalgebra of some powerset Boolean algebra.

For the second construction, the elements of  $M[B]_\omega$  are sequences of the form  $\langle m_1, \dots, m_n; p_1, \dots, p_n \rangle$  for some positive integer  $n$ , where  $m_1, \dots, m_n \in M$  and  $\{p_1, \dots, p_n\}$  is a partition of  $B$ . Each relation  $R$  on  $M$  can be extended to a relation on  $M[B]_\omega$  by defining

$$M[B]_\omega \models R(\langle m_1, \dots, m_n; p_1, \dots, p_n \rangle, \langle \tilde{m}_1, \dots, \tilde{m}_k; \tilde{p}_1, \dots, \tilde{p}_k \rangle)$$

iff for each  $i < n$  and  $j < k$ , if  $p_i \wedge \tilde{p}_j \neq 0$ , then

$$M \models R(m_i, \tilde{m}_j).$$

Algebraic operations can be defined in a similar way. Then  $f = \{(p_1, m_1), \dots, (p_n, m_n)\} \in M[B]_\omega$  as described in the first construction above, and conversely, if  $f \in M[B]_\omega$  as described in the first construction, then  $\langle f(p_1), \dots, f(p_n); p_1, \dots, p_n \rangle \in M[B]_\omega$  as described in the second construction. Hence there is a natural isomorphism between the models resulting from two constructions.

### 0.3 Basic algebraic aspects of Boolean powers

The following basic algebraic aspects of Boolean powers can be found in Burris [1975] and Ash [1975]. In the event that outlines of proofs are given, the original definition of a Boolean power as given by Foster [1953] will be used.

#### Theorem 0.3.1 :

*Suppose that  $M$  and  $M_j, j \in J$ , are models,  $B, \tilde{B}$  and  $B_i, i \in I$  are Boolean algebras, and  $2$  is the 2-element Boolean algebra. Then*

- (a)  $M[B] = M[B]_\omega$  if  $M$  is finite or  $B$  is finite,

- (b)  $M[2] = M[2]_\omega \cong M$ ,
- (c)  $M[\Pi\{B_i : i \in I\}] \cong \Pi\{M[B_i] : i \in I\}$  where  $B_i$  is complete for each  $i \in I$  if  $M$  is infinite,
- (d)  $M[B_1 \times \cdots \times B_n]_\omega \cong M[B_1]_\omega \times \cdots \times M[B_n]_\omega$ ,
- (e)  $2[B] = 2[B]_\omega \cong B$ ,
- (f)  $(\Pi\{M_j : j \in J\})[B] \cong \Pi\{M_j[B] : j \in J\}$  where  $B$  is complete if  $M_j$  is infinite for any  $j \in J$ ,
- (g)  $(M_1 \times \cdots \times M_m)[B]_\omega \cong M_1[B]_\omega \times \cdots \times M_m[B]_\omega$ ,
- (h)  $B[\tilde{B}]_\omega \cong \tilde{B}[B]_\omega$ .

**Proof :**

(a) is clear from the definition of  $M[B]$ . To prove (b), define  $f^m \in M[2]$  by

$$f^m(x) = \begin{cases} 1, & x = m, \\ 0, & \text{otherwise.} \end{cases}$$

Then the function  $H : M \rightarrow M[2]$  defined by

$$H(m) = f^m$$

is an isomorphism. To prove (c), it is sufficient to show that the function  $H : M[\Pi\{B_i : i \in I\}] \rightarrow \Pi\{M[B_i] : i \in I\}$  defined by

$$H(f)(i)(m) = f(m)(i)$$

is an isomorphism. The proof of (d) is similar. (e) holds since the function  $H : 2[B] \rightarrow B$  defined by

$$H(f) = f(1)$$

is an isomorphism. (f) holds since the function  $H : (\Pi\{M_j : j \in J\})[B] \rightarrow \Pi\{M_j[B] : j \in J\}$  defined by

$$H(f)(\langle m_j : j \in J \rangle)(i) = \vee \{f(\langle m_j : j \in J \rangle) : m_i \text{ is fixed, } m_j \in M_j\}$$

is an isomorphism. (g) follows in a similar way. To prove (h), it is sufficient to note that Quackenbush [1972] has shown that  $B[\tilde{B}]_\omega \cong B \oplus \tilde{B}$ , the free product of  $B$  and  $\tilde{B}$ , and the result follows by commutativity of the free-product. Alternatively, suppose that  $f \in B[\tilde{B}]_\omega$  such that

$$f(b_i) = \begin{cases} \tilde{b}_i, & 1 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Define  $\tilde{f} : \tilde{B} \rightarrow B$  by

$$\tilde{f}(\tilde{b}) = \begin{cases} b_{i_1} \wedge \cdots \wedge b_{i_m} \wedge b_{j_1}^c \wedge \cdots \wedge b_{j_k}^c, & \text{where} \\ \tilde{b} = \tilde{b}_{i_1} \vee \cdots \vee \tilde{b}_{i_m}, & \text{for some } \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\} \\ \text{and } \{j_1, \dots, j_k\} = \{1, \dots, n\} - \{i_1, \dots, i_m\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\tilde{f} \in \tilde{B}[B]_\omega$ , and the function  $H : B[\tilde{B}]_\omega \rightarrow \tilde{B}[B]_\omega$  defined by

$$H(f) = \tilde{f}$$

is an isomorphism. □

The following result has already been noted by Ouwehand and Rose [1998] (see Remark 0.2.4.11(c)) in their approach to Boolean powers as the direct limits of direct powers, since the powerset Boolean algebra  $2^I$  has a finest partition consisting of all singletons. It is also, however, a direct consequence of Theorem 0.3.1 (c), (b), and it is thus included for the sake of continuity.

**Corollary 0.3.2 :**

*Suppose that  $M$  is a model and  $I$  is a set. Then*

$$M[2^I] \cong M^I.$$

The following result shows that a finitely iterated bounded Boolean power is isomorphic to a bounded Boolean power. This result is to be expected, since bounded Boolean powers can be constructed as submodels of direct powers. Burris [1975] discusses the problem in the case that  $M$  is an algebra. Firstly, he considers the algebra  $\overline{M}$  in which all finitary functions are fundamental operations. Next, he points out that every bounded subdirect power of  $\overline{M}$  i.e. a subdirect power of  $\overline{M}$  such that each function has a finite range, is isomorphic to a bounded Boolean power of  $\overline{M}$ . Since finitely iterated bounded subdirect powers are isomorphic to bounded subdirect powers, it follows that  $(\overline{M}[B]_\omega)[\tilde{B}]_\omega \cong \overline{M}[\hat{B}]_\omega$ , where  $B, \tilde{B}$  and  $\hat{B}$  are Boolean algebras. If  $\overline{M}$  is non-trivial, then by taking a two-element reduct isomorphic to 2, it follows that  $(2[B]_\omega)[\tilde{B}]_\omega \cong 2[\hat{B}]_\omega$ , and thus  $B[\tilde{B}]_\omega \cong \hat{B}$ , which gives  $(\overline{M}[B]_\omega)[\tilde{B}]_\omega \cong \overline{M}[B[\hat{B}]_\omega]$ . The result is obtained by observing that for any algebra  $M$  there exists a model  $\overline{M}$ , which has

all the finitary functions as fundamental operations, and is such that  $M$  is a reduct of  $\overline{M}$ .

**Theorem 0.3.3 [S. Burris 1975] :**

*Suppose that  $M$  is a model, and  $B$  and  $\tilde{B}$  are Boolean algebras. Then*

$$(M[B]_\omega)[\tilde{B}]_\omega \cong M[B[\tilde{B}]_\omega]_\omega.$$

**Proof :**

Suppose that  $f \in (M[B]_\omega)[\tilde{B}]_\omega$ . Define  $\tilde{f} : M \rightarrow B[\tilde{B}]_\omega$  by

$$\tilde{f}(m)(b) = \vee \{f(g) : g \in M[B]_\omega, g(m) = b\}.$$

Then  $\tilde{f} \in M[B[\tilde{B}]_\omega]_\omega$  and the function  $H : (M[B]_\omega)[\tilde{B}]_\omega \rightarrow M[B[\tilde{B}]_\omega]_\omega$  defined by

$$H(f) = \tilde{f}$$

is an isomorphism. □

**Corollary 0.3.4 [S. Burris 1975] :**

*Suppose that  $M$  is a model, and  $B$  and  $\tilde{B}$  are Boolean algebras. Then*

$$(M[B]_\omega)[\tilde{B}]_\omega \cong (M[\tilde{B}]_\omega)[B]_\omega \cong M[B \oplus \tilde{B}]_\omega.$$

**Proof :**

The result follows by Theorem 0.3.3 and Quackenbush's [1972] result which states that  $B \oplus \tilde{B} \cong B[\tilde{B}]_\omega$ . □

**Corollary 0.3.5 [S. Burris 1975] :**

*Suppose that  $B$ ,  $\tilde{B}$  and  $\hat{B}$  are Boolean algebras and  $n \in \omega$ . Then*

- (a)  $B \oplus (\tilde{B} \times \hat{B}) \cong (B \oplus \tilde{B}) \times (B \oplus \hat{B})$ ,
- (b)  $B \oplus (\tilde{B})^n \cong (B \oplus \tilde{B})^n$ ,
- (c)  $B \oplus 2^n \cong B^n$ .

The following definition is due to Burris [1975].

**Definition 0.3.6 [S. Burris 1975] :**

Suppose that  $M$  is a model, and  $B$  and  $\tilde{B}$  are Boolean algebras.  $M$  is  $\mathcal{B}$ -separating if  $M[B] \cong M[\tilde{B}]$  implies that  $B \cong \tilde{B}$ .

It is clear that a  $\mathcal{B}$ -separating model  $M$  must have at least two elements. Burris [1975] focuses his investigation specifically on  $\mathcal{B}$ -separating algebras, and his main result on this topic is the description of conditions on a simple algebra which will ensure that it is  $\mathcal{B}$ -separating. According to Burris, the study of  $\mathcal{B}$ -separating algebras was motivated by the following : In 1953 Kinoshita gave examples of countable Boolean algebras  $B, \tilde{B}$  and  $\hat{B}$  such that  $B \cong B \times \tilde{B} \times \hat{B}$ , but  $B \not\cong B \times \tilde{B}$ . In 1957 Hanf reinforced the ideas of Kinoshita by describing two countable Boolean algebras  $B$  and  $\tilde{B}$  such that  $B \cong B \times \tilde{B} \times \tilde{B}$ , but  $B \not\cong B \times \tilde{B}$ . Tarski observed that this implies the existence of two countable Boolean algebras  $B$  and  $\tilde{B}$  such that  $B \times B \cong \tilde{B} \times \tilde{B}$ , but  $B \not\cong \tilde{B}$ . Jónsson and Tarski subsequently realised that if  $M$  is an algebra such that  $M[B] \cong M[\tilde{B}]$  implies that  $B \cong \tilde{B}$  for Boolean algebras  $B$  and  $\tilde{B}$ , then similar results for direct products for the class of bounded Boolean powers of  $M$  could be obtained.

The following theorem of Burris [1975] shows two applications of  $\mathcal{B}$ -separating algebras.

**Theorem 0.3.7 [S. Burris 1975] :**

Suppose that  $\mathcal{K}$  is a class of algebras closed under bounded Boolean powers and that  $\mathcal{K}$  contains a  $\mathcal{B}$ -separating algebra  $M$ . Then

- (a) There exist algebras  $\tilde{M}, \hat{M} \in \mathcal{K}$  such that  $(\tilde{M})^2 \cong (\hat{M})^2$ , but  $\tilde{M} \not\cong \hat{M}$ .
- (b)  $\mathcal{K}$  has at least  $2^\lambda$  isomorphism types of algebras of power  $\lambda$ , where  $\lambda \geq |M|$ .

**Proof :**

To prove (a), by Tarski's observation there exist countable Boolean algebras  $B$  and  $\tilde{B}$  such that  $B \times B \cong \tilde{B} \times \tilde{B}$ , but  $B \not\cong \tilde{B}$ . Hence  $M[B \times B]_\omega \cong M[\tilde{B} \times \tilde{B}]_\omega$ , and by Theorem 0.3.1(d), it follows that  $(M[B]_\omega)^2 \cong (M[\tilde{B}]_\omega)^2$ , and since  $M$  is  $\mathcal{B}$ -separating, the result follows. (b) holds since it should be noted that Boolean algebras of power  $\lambda$ ,  $\lambda \geq \aleph_0$ , have  $2^\lambda$  isomorphism types.  $\square$

Burris [1975] quotes Tarski's example of a  $\mathcal{B}$ -separating algebra, namely the additive semigroup  $\langle \omega, + \rangle$ . Tarski generalised this result for the class of countable

centerless indecomposable algebras. Burris furthermore provides an example of an algebra  $M$  such that there exists a Boolean algebra  $B$  such that  $B$  is definable by a first-order formula in  $M[B]_\omega$ , and hence  $M$  is  $\mathcal{B}$ -separating. Finally, he states the following sufficiency theorem for  $\mathcal{B}$ -separating algebras which is based on congruence lattices and which yields  $\mathcal{B}$ -separating algebras which are not obtainable using Jónsson's generalisation.

**Theorem 0.3.8 [S. Burris 1975] :**

*Suppose that  $S$  is a simple algebra such that*

- (a) *the congruence lattice of  $S^n$  is modular for  $n \in \omega$ ,*
- (b) *the congruence lattice of  $S^2$  is isomorphic to the square of the congruence lattice of  $S$ .*

*Then  $S$  is  $\mathcal{B}$ -separating.*

Although the model-theoretic aspects of Boolean powers have not yet been discussed, it is not inappropriate to state the following problem which Burris [1975] poses:

"If  $M$  is  $\mathcal{B}$ -separating and  $M[B]_\omega \equiv M[\tilde{B}]_\omega$ , does it follow that  $B \equiv \tilde{B}$ ?"

Garavaglia and Plotkin [1984] provide a negative answer to Burris' problem. They introduce their discussion with the following definition.

**Definition 0.3.9 [S. Garavaglia and J.M. Plotkin 1984] :**

Suppose that  $M$  is a model, and  $B$  and  $\tilde{B}$  are Boolean algebras.  $M$  is *weakly separating* if  $M[B]_\omega \equiv M[\tilde{B}]_\omega$  implies that  $B \equiv \tilde{B}$ .

**Theorem 0.3.10 [S. Garavaglia and J.M. Plotkin 1984] :**

*Every chain with at least two elements is  $\mathcal{B}$ -separating.*

A first-order sentence  $\phi$  is a *factor sentence* if for all models  $M$  and  $\tilde{M}$ ,  $M \times \tilde{M} \models \phi$  implies that  $M \models \phi$  and  $\tilde{M} \models \phi$ . By using a result of Galvin (quoted in Garavaglia and Plotkin [1984]) which states that if for every factor sentence  $\phi$ ,  $M \models \phi$  iff  $\tilde{M} \models \phi$ , then  $M \equiv \tilde{M}$ , Garavaglia and Plotkin prove the following theorem.

**Theorem 0.3.11 [S. Garavaglia and J.M. Plotkin 1984] :**

*Every chain with at least two elements is weakly separating.*

Garavaglia and Plotkin then proceed to construct a model which is  $\mathcal{B}$ -separating but not weakly separating.



**Lemma 0.3.12 [S. Garavaglia and J.M. Plotkin 1984] :**

*Suppose that  $L$  is the language obtained by extending the language of Boolean algebras to include countably many unary relation symbols  $\{\bar{R}_n : n \in \omega\}$ . Then there exists a countable model  $M = \langle B, \{R_n : n \in \omega\}, 0, 1 \rangle$  of  $L$  such that*

- (a)  *$B$  and  $R_n$ , for all  $n \in \omega$ , are atomless Boolean algebras,  $\{R_n : n \in \omega\}$  is a descending chain of subalgebras of  $B$  and*

$$\cap \{R_n : n \in \omega\} = \{0, 1\},$$

- (b)  $M \times M \equiv M$ .

**Theorem 0.3.13 [S. Garavaglia and J.M. Plotkin 1984] :**

*Suppose that  $M$  is a model as described in Lemma 0.3.12. Then  $M$  is  $\mathcal{B}$ -separating but not weakly separating.*

**Proof :**

By Lemma 0.3.12(a),

$$\{f \in M[B]_\omega : M[B]_\omega \models \bar{R}_n(f) \text{ for all } n \in \omega\} = \{f \in M[B]_\omega : \text{range } f \subseteq \{0, 1\}\},$$

which is isomorphic to  $B$ . Thus  $M[B]_\omega \cong M[\tilde{B}]_\omega$  implies that  $B \cong \tilde{B}$ , and hence  $M$  is  $\mathcal{B}$ -separating. However, by Lemma 0.3.12(b) it follows that  $M$  is not weakly separating for even finite Boolean algebras since

$$M[2^{2^n}]_\omega \cong (M[2]_\omega)^{2^n} \cong (M)^{2^n} \equiv M^n \cong (M[2]_\omega)^n \cong M[2^n]_\omega,$$

but  $2^{2^n} \not\equiv 2^n$ . □

## 0.4 Model-theoretic aspects of Boolean powers

In this section the Boolean power  $M[B]$  and the more general  $\kappa$ -bounded Boolean power  $M[B]_\kappa$  will be discussed from the perspective of Boolean-valued models. Extensive work on the fundamental model-theoretic aspects has been done by Mansfield [1971] and Ouwehand and Rose [1998].

The discussion commences with a detailed account of the results of Mansfield, who bases his proofs on the original definition of the Boolean power of Foster [1953].

Suppose that  $M$  is a model and  $B$  is a complete Boolean algebra. Suppose that  $f_1, \dots, f_n \in M[B]$ . For each relation  $R$  on  $M$  ( $R$  is extended to a relation on  $M[B]$ ), the *Boolean value*  $\|R(f_1, \dots, f_n)\|$  can be defined by

$$\|R(f_1, \dots, f_n)\| = \vee \{f_1(m) \wedge \dots \wedge f_n(m_n) : M \models R(m_1, \dots, m_n)\}.$$

Thus

$$\begin{aligned} M[B] \models R(f_1, \dots, f_n) &\text{ iff } \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models R(m_1, \dots, m_n)\} = 1 \\ &\text{ iff } \|R(f_1, \dots, f_n)\| = 1. \end{aligned}$$

The definition of the Boolean truth valuation  $\|\cdot\|$  can be extended to arbitrary first-order formulas by defining

- (a)  $\|\varphi \vee \psi\| = \|\varphi\| \vee \|\psi\|,$
- (b)  $\|\neg \varphi\| = \|\varphi\|^c,$
- (c)  $\|\exists x \varphi(x)\| = \vee \{\|\varphi(f)\| : f \in M[B]\}.$

**Remark 0.4.1 :**

The equality relation on  $M$  is extended to a relation on  $M[B]$  in the same way as any other relation on  $M$ , namely

$$\begin{aligned} \|f = g\| &= \vee \{f(m) \wedge g(\tilde{m}) : m = \tilde{m}\} \\ &= \vee \{f(m) \wedge g(m) : m \in M\}. \end{aligned}$$

Thus if  $\|f = g\| = 1$ , then  $f = g$  for this manner of extending relations on  $M$  to relations on  $M[B]$ .

**Theorem 0.4.2 [R. Mansfield 1971] :**

*Suppose that  $M$  is model and  $B$  is a complete Boolean algebra. Then for  $f_1, \dots, f_n \in M[B]$  and any formula  $\theta$  in the language of  $M$ ,*

$$\|\theta(f_1, \dots, f_n)\| = \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \theta(m_1, \dots, m_n)\}$$

**Proof :**

If  $\theta$  is atomic, then the result follows by definition of the Boolean value.

If  $\theta = \varphi \vee \psi$ , then

$$\begin{aligned}
\|\theta(f_1, \dots, f_n)\| &= \|\varphi(f_1, \dots, f_n) \vee \psi(f_1, \dots, f_n)\| \\
&= \|\varphi(f_1, \dots, f_n)\| \vee \|\psi(f_1, \dots, f_n)\| \\
&= \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(m_1, \dots, m_n)\} \vee \\
&\quad \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \psi(m_1, \dots, m_n)\} \\
&= \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(m_1, \dots, m_n) \vee \psi(m_1, \dots, m_n)\}.
\end{aligned}$$

If  $\theta = \neg \varphi$ , then

$$\begin{aligned}
\|\theta(f_1, \dots, f_n)\| &= \|\neg \varphi(f_1, \dots, f_n)\| \\
&= \|\varphi(f_1, \dots, f_n)\|^c \\
&= (\vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(m_1, \dots, m_n)\})^c \\
&= \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \neg \varphi(m_1, \dots, m_n)\}, \text{ since} \\
&\quad \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : m_1, \dots, m_n \in M\} \text{ is a partition of } B.
\end{aligned}$$

Finally, if  $\theta = \exists x \varphi$  then

$$\begin{aligned}
\|\theta(f_1, \dots, f_n)\| &= \|\exists x \varphi(x, f_1, \dots, f_n)\| \\
&= \vee \{\|\varphi(g, f_1, \dots, f_n)\| : g \in M[B]\} \\
&= \vee \{\vee \{g(\tilde{m}) \wedge f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(\tilde{m}, m_1, \dots, m_n)\} : g \in M[B]\} \\
&= \vee \{\vee \{g(\tilde{m}) \wedge f_1(m_1) \wedge \dots \wedge f_n(m_n) : g \in M[B]\} : M \models \varphi(\tilde{m}, m_1, \dots, m_n)\}.
\end{aligned}$$

For each fixed  $m \in M$ , define a function  $f^m : M \rightarrow B$  by

$$f^m(x) = \begin{cases} 1, & x = m, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $f^m \in M[B]$  and

$$\begin{aligned}
\|\theta(f_1, \dots, f_n)\| &\geq \vee \{f^{\tilde{m}}(\tilde{m}) \wedge f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(\tilde{m}, m_1, \dots, m_n)\} \\
&\geq \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(\tilde{m}, m_1, \dots, m_n)\}.
\end{aligned}$$

Now since

$$\vee \{g(\tilde{m}) \wedge f_1(m_1) \wedge \dots \wedge f_n(m_n) : g \in M[B]\} \leq f_1(m_1) \wedge \dots \wedge f_n(m_n)$$

as well, it follows that

$$\begin{aligned}\|\theta(f_1, \dots, f_n)\| &= \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \varphi(\tilde{m}, m_1, \dots, m_n)\} \\ &= \vee \{f_1(m_1) \wedge \dots \wedge f_n(m_n) : M \models \exists x \varphi(x, m_1, \dots, m_n)\}. \square\end{aligned}$$

**Remarks 0.4.3 :**

(a) Despite the fact that the equality relation has properties which are not applicable to relations in general, it follows by Theorem 0.4.2 that the equality axioms are all valid i.e.

- (i)  $\|f = f\| = 1$ ,
- (ii)  $\|f = g\| = \|g = f\|$ ,
- (iii)  $\|f = g\| \wedge \|g = h\| \leq \|f = h\|$ ,
- (iv)  $\|f = g\| \wedge \|\varphi(g)\| \leq \|\varphi(f)\|$ .

These axioms are useful for simplifying complex expressions, as will be shown in later chapters.

(b) Another observation is that  $\|f = f^m\| = f(m)$ . According to Mansfield [1971], “this lies at the heart of the intuitive motivation for  $M[B]$ ”. The condition that for  $f \in M[B]$ ,  $\vee \{f(m) : m \in M\} = 1$  implies that there exists  $m \in M$  such that  $f = f^m \in M[2] \cong M$ . Thus, Mansfield continues, to obtain  $M[B]$ , “we start with the set  $M$  and add to it all those abstract objects  $x$  with  $\|x \in M\| = 1$ ”. The following theorem shows that all such elements have indeed been added to  $M$  to obtain  $M[B]$ .

**Theorem 0.4.4 [R. Mansfield 1971] :**

*Suppose that  $\{b_i : i \in I\} \subseteq B$  is a pairwise disjoint set and  $\{f_i : i \in I\} \subseteq M[B]$ . Then there exists  $f \in M[B]$  such that  $\|f = f_i\| \geq b_i$  for each  $i \in I$ . If, in addition,  $\vee \{b_i : i \in I\} = 1$ , then  $f$  is unique and will be written  $f = \Sigma\{b_i.f_i : i \in I\}$ .*

**Proof :**

It can be assumed that  $\vee \{b_i : i \in I\} = 1$ , for otherwise a new index set  $J$  could be defined by

$$J = I \cup \{I\}$$

and  $f_I$  could be any element of  $M[B]$ .

Define a function  $f : M \rightarrow B$  by

$$f(m) = \vee \{b_i \wedge f_i(m) : i \in I\}.$$

$f(m) \wedge f(n) = 0$  if  $m \neq n$ :

$$\begin{aligned} f(m) \wedge f(n) &= \vee\{b_i \wedge f_i(m) : i \in I\} \wedge \vee\{b_j \wedge f_j(n) : j \in I\} \\ &= \vee\{\vee\{b_i \wedge b_j \wedge f_i(m) \wedge f_j(n) : j \in I\} : i \in I\}. \end{aligned}$$

Since  $b_i \wedge b_j = 0$  if  $i \neq j$ , and  $f_i(m) \wedge f_j(n) = 0$  if  $i = j$ , it follows that  $f(m) \wedge f(n) = 0$ .

$\vee\{f(m) : m \in M\} = 1$  :

$$\begin{aligned} \vee\{f(m) : m \in M\} &= \vee\{\vee\{b_i \wedge f_i(m) : i \in I\} : m \in M\} \\ &= \vee\{\vee\{b_i \wedge f_i(m) : m \in M\} : i \in I\} \\ &= \vee\{b_i : i \in I\} = 1. \end{aligned}$$

Hence  $f \in M[B]$ .

Now

$$\begin{aligned} b_i \wedge \|f = f_i\| &= b_i \wedge \vee\{f(m) \wedge f_i(m) : m \in M\} \\ &= \vee\{\vee\{b_i \wedge b_j \wedge f_i(m) \wedge f_j(m) : j \in J\} : m \in M\} \\ &= \vee\{b_i \wedge f_i(m) : m \in M\} = b_i \end{aligned}$$

i.e.  $\|f = f_i\| \geq b_i$ .

To show that  $f$  is unique, suppose that there exists  $g \in M[B]$  such that  $\|g = f_i\| \geq b_i$  for each  $i \in I$ . Then

$$\begin{aligned} \|f = g\| &\geq \|f = f_i\| \wedge \|f_i = g\| \\ &\geq b_i, \quad \text{for each } i \in I, \end{aligned}$$

and hence

$$\|f = g\| \geq \vee\{b_i : i \in I\} = 1.$$

□

**Remarks 0.4.5 :**

- (a) The set  $\{f(m) : m \in M\}$  is pairwise disjoint and  $\vee\{f(m) : m \in M\} = 1$ . By an earlier observation (Remark 0.4.3(b)),  $\|f = f^m\| = f(m)$ , and hence it follows by Theorem 0.4.4 that

$$f = \Sigma\{f(m).f^m : m \in M\}.$$

Furthermore, by identifying  $f^m$  with  $m$  for each  $m \in M$ ,

$$f = \Sigma\{f(m).m : m \in M\}.$$

(b) If  $f = \Sigma\{b_i.f^{m_i} : i \in I\}$ , then

$$\begin{aligned} f(m) &= (\Sigma\{b_i.f^{m_i} : i \in I\})(m) \\ &= \vee\{b_i \wedge f^{m_i}(m) : i \in I\} \text{ by definition of } f(m) \\ &= \vee\{b_i : m_i = m\}. \end{aligned}$$

**Theorem 0.4.6 [R. Mansfield 1971] :**

*Suppose that  $M$  is a model and  $B$  is a complete Boolean algebra. Then for  $f_1, \dots, f_n \in M[B]$  and  $\exists x\varphi$  a formula in the language of  $M$ , there exists  $g \in M[B]$  such that*

$$\|\exists x\varphi(x, f_1, \dots, f_n)\| = \|\varphi(g, f_1, \dots, f_n)\|.$$

**Proof :**

Suppose that  $\langle g_\alpha : \alpha < \lambda \rangle$  is a well-ordering of  $M[B]$ . Then

$$\|\exists x\varphi(x, f_1, \dots, f_n)\| = \vee\{\|\varphi(g_\alpha, f_1, \dots, f_n)\| : \alpha < \lambda\}.$$

Now suppose that

$$b_\alpha = \|\varphi(g_\alpha, f_1, \dots, f_n)\| \wedge (\vee\{\|\varphi(g_\gamma, f_1, \dots, f_n)\| : \gamma < \alpha\})^c.$$

Then if  $\alpha \neq \beta$ ,

$$\begin{aligned} b_\alpha \wedge b_\beta &= \|\varphi(g_\alpha, f_1, \dots, f_n)\| \wedge (\vee\{\|\varphi(g_\gamma, f_1, \dots, f_n)\| : \gamma < \alpha\})^c \\ &\quad \wedge \|\varphi(g_\beta, f_1, \dots, f_n)\| \wedge (\vee\{\|\varphi(g_\delta, f_1, \dots, f_n)\| : \delta < \beta\})^c \\ &= \|\varphi(g_\alpha, f_1, \dots, f_n)\| \wedge \wedge\{\|\varphi(g_\gamma, f_1, \dots, f_n)\|^c : \gamma < \alpha\} \\ &\quad \wedge \|\varphi(g_\beta, f_1, \dots, f_n)\| \wedge \wedge\{\|\varphi(g_\delta, f_1, \dots, f_n)\|^c : \delta < \beta\} \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} b_\alpha &= \|\varphi(g_\alpha, f_1, \dots, f_n)\| \wedge (\vee\{\|\varphi(g_\gamma, f_1, \dots, f_n)\| : \gamma < \alpha\})^c \\ &\geq \|\varphi(g_\alpha, f_1, \dots, f_n)\| \wedge (\vee\{\|\varphi(g_\delta, f_1, \dots, f_n)\| : \delta < \lambda\})^c, \end{aligned}$$

and hence

$$\begin{aligned} \vee \{b_\alpha : \alpha < \lambda\} &= \vee \{\|\varphi(g_\alpha, f_1, \dots, f_n)\| : \alpha < \lambda\} \\ &= \|\exists x \varphi(x, f_1, \dots, f_n)\|. \end{aligned}$$

By Theorem 0.4.4, there exists  $g \in M[B]$  such that  $\|g = g_\alpha\| \geq b_\alpha$ . Then

$$\begin{aligned} \|\varphi(g, f_1, \dots, f_n)\| &\geq \|\varphi(g_\alpha, f_1, \dots, f_n)\| \wedge \|g = g_\alpha\| \\ &\geq b_\alpha, \quad \text{for each } \alpha < \lambda, \end{aligned}$$

and hence

$$\begin{aligned} \|\varphi(g, f_1, \dots, f_n)\| &\geq \vee \{b_\alpha : \alpha < \lambda\} \\ &= \|\exists x \varphi(x, f_1, \dots, f_n)\|. \end{aligned}$$

□

**Remarks 0.4.7 :**

If  $B$  is not complete, then the definition of  $\|\exists x \varphi\|$  with respect to  $M[B]$  appears to be problematic because  $\vee \{\|\varphi(f)\| : f \in M[B]\}$  may not exist in  $B$ . This is the case for the  $\kappa$ -bounded Boolean power  $M[B]_\kappa$  ( $B$  is  $\kappa$ -complete). As the basis for getting around this problem, Ouwehand and Rose [1998] note that if  $B$  is complete, then the definition of  $\|\exists x \varphi\|$  is valid, and if  $C$  is a complete subalgebra of  $B$ , then  $M[C]$  is a submodel of  $M[B]$  and the Boolean values with respect to  $M[C]$  and  $M[B]$  coincide. In order to obtain analogous results to Theorem 0.4.2 and Theorem 0.4.6 for  $\kappa$ -bounded Boolean powers, they draw attention to the fact that every Boolean algebra  $B$  has a completion  $\overline{B}$  in which it is densely embedded. In particular, every partition of  $\overline{B}$  has a refinement in  $B$ . Hence if  $\varphi$  is a first-order formula and  $f_1, \dots, f_n \in M[B]_\kappa$ , then the Boolean values with respect to  $M[\overline{B}]$  and  $M[B]_\kappa$  coincide, on condition that the latter Boolean value exists. Ouwehand and Rose of course use their construction of  $M[B]_\kappa$  as a direct limit of direct powers to prove these results. Note that to distinguish between the different Boolean values,  $\|\cdot\|_{M[B]_\kappa}$  and  $\|\cdot\|_{M[\overline{B}]}$  will be used to denote the Boolean values with respect to  $M[B]_\kappa$  and  $M[\overline{B}]$  respectively.

**Theorem 0.4.8 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $M$  is a model and  $B$  is a  $\kappa$ -complete Boolean algebra.*

- (a) *If  $f_1, \dots, f_n \in M[B]_\kappa$ ,  $P$  is a common domain of  $f_1, \dots, f_n$ , and  $\varphi$  is a formula in the language of  $M$ , then*

$$\|\varphi(f_1, \dots, f_n)\| = \vee \{p \in P : M \models \varphi(f_1(p), \dots, f_n(p))\}.$$

- (b) If  $f_1, \dots, f_n \in M[B]_\kappa$  and  $\exists x\varphi$  is a formula in the language of  $M$ , then there exists  $g \in M[B]_\kappa$  such that

$$\|\exists x\varphi(x, f_1, \dots, f_n)\| = \|\varphi(g, f_1, \dots, f_n)\|.$$

**Proof :**

Suppose that  $\overline{B}$  is the completion of  $B$ . It will be shown by simultaneous induction that  $\|\varphi(f_1, \dots, f_n)\|_{M[B]_\kappa}$  always exists and equals  $\vee\{p \in P : M \models \varphi(f_1(p), \dots, f_n(p))\}$ . In the process, it will become evident that there exists  $g \in M[B]_\kappa$  such that  $\|\exists x\varphi(x, f_1, \dots, f_n)\| = \|\varphi(g, f_1, \dots, f_n)\|$ . With the exception of formulas of the form  $\exists x\varphi$ , it is a straightforward matter to prove (a) for all atomic formulas. By Theorem 0.4.6, there exists  $g \in M[\overline{B}]$  such that

$$\|\exists x\varphi(x, f_1, \dots, f_n)\|_{M[\overline{B}]} = \|\varphi(g, f_1, \dots, f_n)\|_{M[\overline{B}]}.$$

Suppose that  $Q$  is a domain of  $g$ . It can be assumed that  $Q$  is a partition of  $B$  and that  $Q$  is finer than  $P$ . By Theorem 0.4.2, it follows that

$$\|\varphi(g, f_1, \dots, f_n)\|_{M[\overline{B}]} = \vee\{q \in Q : M \models \varphi(g(q), f_1(q), \dots, f_n(q))\}.$$

Define a function  $\tilde{g} : P \rightarrow M$  by

$$\tilde{g}(p) = \begin{cases} g(q), \text{ for some } q \leq p \in P \text{ such that} \\ M \models \varphi(g(q), f_1(q), \dots, f_n(q)), \text{ provided} \\ \text{such } q \text{ exists,} \\ \\ \text{arbitrary, otherwise.} \end{cases}$$

Then

$$\begin{aligned} \|\exists x\varphi(x, f_1, \dots, f_n)\|_{M[\overline{B}]} &= \|\varphi(g, f_1, \dots, f_n)\|_{M[\overline{B}]} \\ &= \vee\{q \in Q : M \models \varphi(g(q), f_1(q), \dots, f_n(q))\} \\ &\leq \vee\{p \in P : M \models \varphi(\tilde{g}(p), f_1(p), \dots, f_n(p))\} \\ &= \|\varphi(\tilde{g}, f_1, \dots, f_n)\|_{M[B]_\kappa}, \text{ which exists by hypothesis} \\ &\leq \|\exists x\varphi(x, f_1, \dots, f_n)\|_{M[\overline{B}]}, \end{aligned}$$

and the result follows.  $\square$



**Corollary 0.4.9 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $M$  is a model,  $B$  is a  $\kappa$ -complete Boolean algebra and  $\tau \leq \kappa$ . If  $f_1, \dots, f_n \in M[B]_\tau$  and  $\varphi$  is a formula in the language of  $M$ , then the Boolean values of  $\varphi(f_1, \dots, f_n)$  with respect to  $M[B]_\tau$  and  $M[B]_\kappa$  coincide.*

**Definition 0.4.10 :**

Suppose that  $M$  is a  $B$ -valued model.  $M$  is *full* if for each formula  $\exists x \varphi(x, \bar{m})$  with parameters  $\bar{m}$  from  $M$ , there exists  $\tilde{m} \in M$  such that  $\|\exists x \varphi(x, \bar{m})\| = \|\varphi(\tilde{m}, \bar{m})\|$ .

Theorem 0.4.6 (Mansfield [1971]) and Theorem 0.4.8(b) (Ouwehand and Rose [1998]) are just statements to the effect that  $M[B]$  and  $M[B]_\kappa$  respectively, are full.

**Definition 0.4.11 :**

Suppose that  $M$  is a  $B$ -valued model.  $M$  *admits glueing over  $B$*  if for every finite disjoint subset  $U$  of  $B$ , and every family  $\{m_u : u \in U\}$  of elements of  $M$  indexed by  $U$ , there exists  $m \in M$  such that for each  $u \in U$ ,  $\|m = m_u\| \geq u$ .

Mansfield [1971] has shown that  $M[B]$  admits glueing over  $B$  (Theorem 0.4.4). The following lemma is the adaptation by Ouwehand and Rose [1998] which shows that  $M[B]_\kappa$  also admits glueing over  $B$ .

**Lemma 0.4.12 [P.Ouwehand and H. Rose 1998] :**

*Suppose that  $\kappa$  is regular. Suppose that  $U \subseteq B$  is a pairwise disjoint set and  $|U| < \kappa$ . Then for any family  $\{f_u : u \in U\} \subseteq M[B]_\kappa$ , there exists  $f \in M[B]_\kappa$  such that  $\|f = f_u\| \geq u$  for each  $u \in U$ .*

**Proof :**

Since  $U$  is disjoint, by extending  $U$  if necessary,  $U$  can be assumed to be a partition of  $B$ . For each  $u \in U$ , suppose that  $P_u \in \mathbb{P}_B^\kappa$  is a domain of  $f_u$ . By taking a common refinement of  $U$  and  $P_u$  if necessary,  $P_u$  can be assumed to be a refinement of  $U$ . Then for each  $u \in U$ , there exists  $X_u \subseteq P_u$  such that

$\forall X_u = u$ . Now define  $P = \cup\{X_u : u \in U\}$ . Clearly  $P \in \mathbb{P}_B^\kappa$ . Define  $f : P \rightarrow M$  such that  $f \upharpoonright P_u = f_u$ . Then

$$\begin{aligned} \|f = f_p\| &= \vee\{p \in P_u : f(p) = f_u(p)\} \\ &\geq \vee\{p \in X_u : f(p) = f_u(p)\} \\ &= u, \text{ as required.} \end{aligned}$$

□

Ouwehand and Rose [1998] present the following version of the Feferman-Vaught Theorem for  $B$ -valued models based on the generalised version of the theorem as discussed in Hodges [1993].

**Theorem 0.4.13 [P.Ouwehand and H. Rose 1998] :**

*Suppose that  $L$  is a first-order language and  $\varphi(\bar{x})$  is a formula of  $L$ . Then there exists an algorithm which computes  $\varphi^b$  and  $\theta_1(\bar{x}), \dots, \theta_n(\bar{x})$  such that*

- (a)  $\varphi^b$  is a formula of the theory of Boolean algebras,
- (b)  $\theta_1(\bar{x}), \dots, \theta_n(\bar{x})$  are formulas of  $L$

*with the following properties: If  $M$  is a  $B$ -valued model that is full and admits glueing over  $B$ , then*

$$M \models \varphi(\bar{m}) \text{ iff } B \models \varphi^b(\|\theta_1(\bar{m})\|, \dots, \|\theta_n(\bar{m})\|).$$

Note that the formulas  $\varphi^b$  and  $\theta_1, \dots, \theta_n$  do not depend on  $M$  or  $B$ , but only on  $\varphi$ .

The following model-theoretic results for bounded Boolean powers can be found in Burris [1975].

**Theorem 0.4.14 :**

*Suppose that  $M$  and  $\widetilde{M}$  are models, and  $B$  and  $\widetilde{B}$  are Boolean algebras.*

- (a) *If  $M \equiv \widetilde{M}$  and  $B \equiv \widetilde{B}$ , then  $M[B]_\omega \equiv \widetilde{M}[\widetilde{B}]_\omega$ .*
- (b) *If  $M \prec \widetilde{M}$  and  $B \prec \widetilde{B}$ , then  $M[B]_\omega \prec \widetilde{M}[\widetilde{B}]_\omega$ .*
- (c) *If  $B$  is complete and  $M \equiv \widetilde{M}$ , then  $M[B] \equiv \widetilde{M}[B]$ . A similar result holds for  $M \prec \widetilde{M}$ .*
- (d) *If  $\mathcal{K}$  is a class of Boolean algebras with  $Th(\mathcal{K})$  decidable, and if  $Th(M)$  is decidable, then  $Th(\{M[B]_\omega : B \in \mathcal{K}\})$  is decidable.*
- (e) *Every bounded Boolean power of  $M$  is elementarily equivalent to a reduced power of  $M$ , and vice versa.*

- (f) *A first-order sentence is preserved under bounded Boolean powers iff it is equivalent to a disjunction of Horn sentences.*
- (g) *An elementary class  $\mathcal{K}$  is closed under bounded Boolean powers iff it is closed under reduced powers iff it is definable by a set of sentences, each of which is a disjunction of Horn sentences.*
- (h) *If  $Th(M)$  and  $Th(B)$  are  $\aleph_0$ -categorical, then  $Th(M[B]_\omega)$  is  $\aleph_0$ -categorical.*
- (i) *If  $M$  is finite and  $\mathcal{B}$ -separating, then  $M[B] \equiv M[\widehat{B}]$  implies that  $B \equiv \widehat{B}$ .*

By applying the Feferman-Vaught theorem for  $\kappa$ -bounded Boolean powers (Theorem 0.4.13), Ouwehand and Rose [1998] obtain the following version of Theorem 0.4.14(a), (b) for  $\kappa$ -bounded Boolean powers.

**Theorem 0.4.15 [ P.Ouwehand and H.Rose 1998] :**

*Suppose that  $M$  and  $\widetilde{M}$  are models, and  $B$  and  $\widetilde{B}$  are  $\kappa$ -complete and  $\tau$ -complete Boolean algebras respectively.*

- (a) *If  $M \equiv \widetilde{M}$  and  $B \equiv \widetilde{B}$ , then  $M[B]_\kappa \equiv \widetilde{M}[\widetilde{B}]_\tau$ .*
- (b) *If  $M \prec \widetilde{M}$  and  $B \prec \widetilde{B}$ , then  $M[B]_\kappa \prec \widetilde{M}[\widetilde{B}]_\tau$ .*

**Proof :**

- (a) For any sentence  $\varphi$ , the formulas  $\theta_1, \dots, \theta_n$  are sentences and their Boolean values are either 0 or 1. Hence

$$\begin{aligned}
 M[B]_\kappa \models \varphi &\text{ iff } B \models \varphi^b(\|\theta_1\|, \dots, \|\theta_n\|) \\
 &\text{ iff } \widetilde{B} \models \varphi^b(\|\theta_1\|, \dots, \|\theta_n\|) \\
 &\text{ iff } \widetilde{M}[\widetilde{B}]_\tau \models \varphi.
 \end{aligned}$$

- (b) can be obtained in a similar way. □

**Corollary 0.4.16 [P. Ouwehand and H. Rose 1998] :**

*Any two infinite direct powers of a model are elementarily equivalent.*

**Proof :**

Any direct power  $M^I \equiv M[\mathcal{P}(I)]$  is a  $\mathcal{P}(I)$ -valued model, and any two infinite powerset algebras are elementarily equivalent by a well-known result of Tarski. (See Monk and Bonnet [1989], vol. 1, section 18.) The result follows from Theorem 0.4.15(a). □

Ouwehand and Rose [1998] make the following observation.

**Remark 0.4.17 :**

Contrary to expectation, any two Boolean powers of a model with respect to an infinite Boolean algebra are not necessarily elementarily equivalent, even though each Boolean power is the direct limit of elementarily equivalent models, namely the infinite direct powers. For suppose that  $B = \mathcal{P}(X)$  and  $\tilde{B}$  is a complete atomless Boolean algebra, and consider the Boolean powers  $B[B]$  and  $B[\tilde{B}]$ . Now  $B[B] = B[\mathcal{P}(X)] \cong B^X$  is an atomic Boolean algebra. However,  $B[\tilde{B}]$  is atomless : Suppose that  $f \in B[\tilde{B}]$  and  $P \in \mathbb{P}_{\tilde{B}}$  is a domain of  $f$ . Since  $\tilde{B}$  is atomless, there exists  $Q \in \mathbb{P}_{\tilde{B}}$  such that for  $p \in P$ ,  $Q = \{q_1, q_2 < p : q_1 \wedge q_2 = 0 \text{ and } q_1 \vee q_2 = p\}$ . Define a function  $g : Q \rightarrow B$  by  $g(q_1) = f(p), g(q_2) = 0$ . Then  $g \in B[\tilde{B}]$  and  $B[\tilde{B}] \models g < f$ . Hence  $B[\tilde{B}]$  is atomless, and  $B[B] \not\equiv B[\tilde{B}]$ .

The following result concludes the discussion on the model-theoretic aspects of Boolean powers.

**Theorem 0.4.18 :**

*Suppose that  $M$  is a model and  $B$  is a  $\kappa$ -complete Boolean algebra for some  $\kappa \geq \aleph_0$ . If  $A$  is any infinite Boolean algebra, then  $M[B]_\kappa$  is embeddable into an ultrapower of  $M[A]_\omega$ .*

**Proof :**

By Theorem 0.4.15(a),  $M[B]_\kappa \equiv M[B]_\omega$ . Since  $A$  is an infinite Boolean algebra,  $B$  is embeddable into an ultrapower of  $A$  i.e. there exist a set  $I$  and an ultrafilter  $F$  on  $\mathcal{P}(I)$  such that  $B$  is embeddable into  $A^I/F$ . Hence  $M[B]_\omega$  is embeddable into  $M[A^I/F]_\omega$ . By a result of Burris [1975] (Theorem 0.5.7 in this thesis),  $M[A^I/F]_\omega \cong M[A^I]_\omega/F$ . By Theorem 0.3.1(d),  $M[A^I]_\omega/F \cong (M[A]_\omega)^I/F$ , and the result follows.

## 0.5 Boolean Ultrapowers

Boolean ultrapowers are, in the words of Mansfield [1971], “serious and useful tools in Model Theory”. The construction of a Boolean ultrapower is completely analogous to the construction of an ordinary ultrapower, the difference being that a Boolean power now assumes the role of the direct power. In other words, an arbitrary complete Boolean algebra is used instead of the customary  $2^I$ .

Suppose that  $M[B]$  is a Boolean power and  $F$  is an ultrafilter on  $B$ . Define an equivalence relation  $\sim$  on  $M[B]$  by

$$f \sim g \text{ iff } \|f = g\| \in F.$$

The equivalence class of  $f$  with respect to  $\sim$  will be denoted by  $f/F$ . The universe of the *Boolean ultrapower*  $M[B]/F$  is defined to be the set

$$\{f/F : f \in M[B]\}.$$

Each relation  $R$  on  $M[B]$  can be extended to a relation on  $M[B]/F$  by defining

$$M[B]/F \models R(f_1/F, \dots, f_n/F) \text{ iff } \|R(f_1, \dots, f_n)\| \in F.$$

Algebraic operations are treated similarly. Thus  $M[B]/F$  is a model of the same type as  $M$ . Note that  $M[B]/F$  is a  $B/F$ -valued model.

**Remark 0.5.1 :**

- (a) If  $F$  is not an ultrafilter, then Ouwehand and Rose [1998] use the term *filtral power* for the model  $M[B]/F$ .
- (b) If  $B$  is  $\kappa$ -complete, then the above-mentioned authors refer to the model  $M[B]_\kappa/F$  as a  $\kappa$ -bounded Boolean ultra-(filtral) power, depending on whether  $F$  is an ultrafilter or not.

Mansfield [1971] proves the following Boolean version of Los's Ultraproduct Theorem.

**Theorem 0.5.2 [R. Mansfield 1971] :**

Suppose that  $M$  is a model and  $F$  is an ultrafilter on a complete Boolean algebra  $B$ . Then if  $f_1, \dots, f_n \in M[B]$  and  $\varphi$  is a formula in the language of  $M$ ,

$$M[B]/F \models \varphi(f_1/F, \dots, f_n/F) \text{ iff } \|\varphi(f_1, \dots, f_n)\| \in F.$$

**Proof :**

If  $\varphi$  is atomic, then the result follows by definition.

If  $\varphi = \theta \vee \psi$ , then

$$\begin{aligned} \|\varphi(f_1, \dots, f_n)\| \in F &\text{ iff } \|\theta(f_1, \dots, f_n) \vee \psi(f_1, \dots, f_n)\| \in F \\ &\text{ iff } \|\theta(f_1, \dots, f_n)\| \in F \text{ or } \|\psi(f_1, \dots, f_n)\| \in F \\ &\text{ iff } M[B]/F \models \theta(f_1/F, \dots, f_n/F) \text{ or } M[B]/F \models \psi(f_1/F, \dots, f_n/F) \\ &\text{ iff } M[B]/F \models \theta(f_1/F, \dots, f_n/F) \vee \psi(f_1/F, \dots, f_n/F) \\ &\text{ iff } M[B]/F \models \varphi(f_1/F, \dots, f_n/F). \end{aligned}$$

If  $\varphi = \ulcorner \psi$ , then

$$\begin{aligned}
\|\varphi(f_1, \dots, f_n)\| \in F &\text{ iff } \|\ulcorner \psi(f_1, \dots, f_n)\| \in F \\
&\text{ iff } \|\psi(f_1, \dots, f_n)\|^c \in F \\
&\text{ iff } \|\psi(f_1, \dots, f_n)\| \notin F \\
&\text{ iff } M[B]/F \not\models \psi(f_1/F, \dots, f_n/F) \\
&\text{ iff } M[B]/F \models \ulcorner \psi(f_1/F, \dots, f_n/F) \\
&\text{ iff } M[B]/F \models \varphi(f_1/F, \dots, f_n/F).
\end{aligned}$$

If  $\varphi = \exists x \psi$ , then

$$\begin{aligned}
\|\varphi(f_1, \dots, f_n)\| \in F &\text{ iff } \|\exists x \psi(x, f_1, \dots, f_n)\| \in F \\
&\text{ iff } \exists g \in M[B] [\|\psi(g, f_1, \dots, f_n)\| \in F] \text{ since } M[B] \text{ is full} \\
&\text{ iff } \exists g \in M[B] [M[B]/F \models \psi(g/F, f_1/F, \dots, f_n/F)] \\
&\text{ iff } M[B]/F \models \exists x \psi(x, f_1/F, \dots, f_n/F) \\
&\text{ iff } M[B]/F \models \varphi(f_1/F, \dots, f_n/F).
\end{aligned}$$

□

**Corollary 0.5.3 :**

*$M[B]/F$  is an elementary extension of  $M$ .*

**Remark 0.5.4 :**

The fact that Boolean powers are full is necessary to prove Theorem 0.5.2. Since  $\kappa$ -bounded Boolean powers are also full, Theorem 0.5.2 also holds for  $\kappa$ -bounded Boolean ultrapowers.

From the above remark, in conjunction with Corollary 0.4.9, Ouwehand and Rose [1998] deduce the following result.

**Theorem 0.5.5 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $\tau$  is an infinite cardinal and  $\tau \leq \kappa$ . Then  $M[B]_\tau/F$  is an elementary submodel of  $M[B]_\kappa/F$ .*

As has been mentioned already, of significance in the work of Ouwehand and Rose [1998] is their approach to ( $\kappa$ -bounded) Boolean powers as direct limits of direct powers. Similarly, they approached ( $\kappa$ -bounded) Boolean ultrapowers as direct limits of ordinary ultrapowers, as is evident in the following lemma.

**Lemma 0.5.6 [P. Ouwehand and H. Rose 1998] :**

- (a) *Suppose that  $B$  is a  $\kappa$ -complete Boolean algebra and  $F$  is a filter on  $B$ . Suppose that  $B$  is the direct limit of  $\{B_P : P \in \mathcal{U}\}$  and  $B$  has the  $\kappa$ -partition cofinality property with respect to  $\{B_P : P \in \mathcal{U}\}$ . Then the  $\kappa$ -bounded filtral power  $M[B]_\kappa/F$  is the direct limit of the  $\kappa$ -bounded filtral powers  $\{M[B_P]_\kappa/F_P : P \in \mathcal{U}\}$  where  $F_P = F \cap B_P$  for each  $P \in \mathcal{U}$ .*
- (b) *Every  $\kappa$ -bounded filtral power  $M[B]_\kappa/F$  is a direct limit of the reduced powers  $\{M^P/F_P : P \in \mathbb{P}_B^\kappa\}$ .*

**Proof :**

(a) follows directly from the definition of the “ $\kappa$ -partition cofinality property”. To prove (b), the result in (a) implies that it is sufficient to show that any  $\kappa$ -complete Boolean algebra  $B$  has the  $\kappa$ -partition cofinality property with respect to  $\{\mathcal{P}(P) : P \in \mathbb{P}_B^\kappa\}$ , where for each  $P \in \mathbb{P}_B^\kappa$ ,  $\mathcal{P}(P)$  is identified with the complete subalgebra of  $B$  under the complete Boolean algebra embedding  $i_P : \mathcal{P}(P) \rightarrow B$  defined by

$$i_P(U) = \bigvee U.$$

Now suppose that  $Q \in \mathbb{P}_B^\kappa$ . Then  $Q$  is a partition of  $\mathcal{P}(Q)$ . By (a), it follows that  $M[B]_\kappa/F$  is the direct limit of  $\{M[\mathcal{P}(P)]_\kappa/F_P : P \in \mathbb{P}_B^\kappa\}$ . Since  $|P| < \kappa$  for each  $P \in \mathbb{P}_B^\kappa$ ,  $M[\mathcal{P}(P)]_\kappa/F_P = M[\mathcal{P}(P)]/F_P \cong M^P/F_P$ , and the result follows.  $\square$

The following results which are relevant to filtral powers can be found in Burris [1975].

**Theorem 0.5.7 :**

*Suppose that  $M$  is a model,  $B$  is a Boolean algebra and  $F$  is a filter on  $B$ . Then*

$$M[B]_\omega/F \cong M[B/F]_\omega.$$

**Theorem 0.5.8 :**

*Suppose that  $M$  is a model,  $I$  is a set and  $F$  is a filter on  $\mathcal{P}(I)$ . Then  $M[\mathcal{P}(I)/F]_\omega$  is isomorphic to an elementary submodel of  $M^I/F$ .*

Using the above results, Ouwehand and Rose [1998] generalise a theorem of Banaschewski and Nelson [1980].

**Theorem 0.5.9 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $M$  is a model,  $B$  is an infinite complete Boolean algebra and  $F$  is a filter on  $B$ . Then*

$$M[B]_\omega/F \prec M[B]/F.$$

**Proof :**

By Lemma 0.5.6(b),  $M[B]/F$  is a direct limit of the reduced powers  $\{M^P/F_P : P \in \mathbb{P}_B^\kappa\}$ . By Theorem 0.5.8,  $M[\mathcal{P}(P)/F_P]_\omega$  is elementarily embeddable into  $M^P/F_P$ . By Theorem 0.5.7,  $M[\mathcal{P}(P)/F_P]_\omega \cong M[\mathcal{P}(P)]_\omega/F_P$ . Hence  $M[\mathcal{P}(P)]_\omega/F_P \prec M^P/F_P$  for each  $P \in \mathbb{P}_B^\kappa$ , and the result follows by taking direct limits on each side.  $\square$

Ouwehand and Rose [1998] then extend Theorem 0.4.14(g) to incorporate elementary classes which are closed under direct limits. An important example of such a class is the amalgamation subclass of an elementary class.

**Theorem 0.5.10 [P. Ouwehand and H. Rose 1998]:**

*Suppose that  $\mathcal{K}$  is an elementary class closed under direct limits.*

- (a) *The following are equivalent:*
  - (i)  $\mathcal{K}$  is closed under finite direct powers,
  - (ii)  $\mathcal{K}$  is closed under  $\kappa$ -bounded Boolean powers for any cardinal  $\kappa$ ,
  - (iii)  $\mathcal{K}$  is closed under reduced powers,
  - (iv)  $\mathcal{K}$  is closed under filtral powers,
  - (v)  $\mathcal{K}$  is defined by a disjunction of Horn sentences.
- (b) *The following are equivalent:*
  - (vi)  $\mathcal{K}$  is closed under finite direct products,
  - (vii)  $\mathcal{K}$  is closed under reduced products and is therefore defined by Horn sentences.

## 0.6 A diversion : Los's Theorem for the $(B, P)$ -product of models

If  $B$  is a complete Boolean algebra, then by Lemma 0.1.29,

$$B \cong \Pi\{B \restriction p : p \in P\},$$

where  $B \restriction p$  is a relative algebra of  $B$  for each  $p \in P \in \mathbb{P}_B$ . Furthermore, if  $M$  is a model, then by Theorem 0.3.1(c),

$$M[B] \cong M[\Pi\{B \restriction p : p \in P\}] \cong \Pi\{M[B \restriction p] : p \in P\}.$$



From the above, the Boolean power  $M[B]$  is isomorphic to the  $(B, P)$ - product  $\Pi\{M[B \restriction p] : p \in P\}$  and this prompts further investigation into the generalised  $(B, P)$ -product  $\Pi\{M_p[B \restriction p] : p \in P\}$  and the  $(B, P)$ -ultraproduct  $\Pi\{M_p[B \restriction p] : p \in P\}/F$ , where  $M_p \neq M$  for each  $p \in P$ .

Suppose that  $B$  is a complete Boolean algebra and  $P \in \mathbb{P}_B$ . For each  $p \in P$ , suppose that  $M_p$  is a model and  $B \restriction p$  is the relative algebra of  $B$  with respect to  $p$ . The universe of the  $(B, P)$ -product  $\Pi\{M_p[B \restriction p] : p \in P\}$  is the set consisting of sequences of the form

$$f = \langle f(p) : p \in P \rangle$$

where  $f(p) \in M_p[B \restriction p]$  for each  $p \in P$ .

(Note that the Boolean power  $M_p[B \restriction p]$  is defined in accordance with the definition of the Boolean power as given by Ouwehand and Rose [1998].)

Now suppose that  $f_1, \dots, f_n \in \Pi\{M_p[B \restriction p] : p \in P\}$  and  $R$  is a relation on  $M_p$  for each  $p \in P$ . The Boolean value  $\|R(f_1, \dots, f_n)\|$  can be defined by

$$\|R(f_1, \dots, f_n)\| = \vee\{p \in P : M_p[B \restriction p] \models R(f_1(p), \dots, f_n(p))\}.$$

The definition of the Boolean truth valuation  $\|\cdot\|$  can be extended to arbitrary first-order formulas in the usual way.

The following lemma is the generalisation of Theorem 0.4.8 for the  $(B, P)$ - product of models.

**Lemma 0.6.1 :**

*Suppose that  $\Pi\{M_p[B \restriction p] : p \in P\}$  is as defined above.*

(a) *If  $f_1, \dots, f_n \in \Pi\{M_p[B \restriction p] : p \in P\}$  and  $\varphi$  is a first-order formula, then*

$$\|\varphi(f_1, \dots, f_n)\| = \vee\{p \in P : M_p[B \restriction p] \models \varphi(f_1(p), \dots, f_n(p))\}.$$

(b) *If  $f_1, \dots, f_n \in \Pi\{M_p[B \restriction p] : p \in P\}$  and  $\exists x \varphi$  is a first-order formula, then there exists  $g \in \Pi\{M_p[B \restriction p] : p \in P\}$  such that*

$$\|\exists x \varphi(x, f_1, \dots, f_n)\| = \|\varphi(g, f_1, \dots, f_n)\|.$$

**Proof :**

(a) follows by induction on the complexity of  $\varphi$ , using the fact that  $M_p[B \restriction p]$  is full for each  $p \in P$  to show that the result is true in the case that  $\varphi$  is an existential formula. (b) follows by defining  $g \in \Pi\{M_p[B \restriction p] : p \in P\}$  by

$$g = \langle g(p) : p \in P \rangle$$

where  $g(p) \in M_p[B \restriction p]$  such that

$$\|\exists x \varphi(x, f_1(p), \dots, f_n(p))\| = \|\varphi(g(p), f_1(p), \dots, f_n(p))\|.$$

□

Now suppose that  $F$  is an ultrafilter on the complete subalgebra of  $B$  which is identified with  $\mathcal{P}(P)$ . Define an equivalence relation  $\sim$  on  $\Pi\{M_p[B \restriction p] : p \in P\}$  by

$$f \sim g \text{ iff } \|f = g\| \in F.$$

Denote the equivalence class of  $f$  with respect to  $\sim$  by  $f/F$ . The universe of the  $(B, P)$ -ultraproduct  $\Pi\{M_p[B \restriction p] : p \in P\}/F$  is the set

$$\{f/F : f \in \Pi\{M_p[B \restriction p] : p \in P\}\}.$$

Each relation  $R$  on  $\Pi\{M_p[B \restriction p] : p \in P\}$  can be extended to a relation on  $\Pi\{M_p[B \restriction p] : p \in P\}/F$  by defining

$$\Pi\{M_p[B \restriction p] : p \in P\}/F \models R(f_1/F, \dots, f_n/F) \text{ iff } \|R(f_1, \dots, f_n)\| \in F.$$

Using the above definition of the  $(B, P)$ -ultraproduct of models, Los's Theorem can now be formulated for  $(B, P)$ -ultraproducts.

**Theorem 0.6.2 :**

*Suppose that  $\Pi\{M_p[B \restriction p] : p \in P\}/F$  is a  $(B, P)$ -ultraproduct as defined above. Then if  $f_1, \dots, f_n \in \Pi\{M_p[B \restriction p] : p \in P\}$  and  $\varphi$  is a first-order formula,*

$$\Pi\{M_p[B \restriction p] : p \in P\}/F \models \varphi(f_1/F, \dots, f_n/F) \text{ iff } \|\varphi(f_1, \dots, f_n)\| \in F.$$

**Proof :**

The proof is similar to that for Boolean ultrapowers (Theorem 0.5.2), except that, in the case of existential formulas, the fact that  $\Pi\{M_p[B \restriction p] : p \in P\}$  is full, is required.  $\square$

**Remarks 0.6.3 :**

(a) If  $M_p = M$  for all  $p \in P$ , then

$$\begin{aligned}\Pi\{M_p[B \restriction p] : p \in P\}/F &= \Pi\{M[B \restriction p] : p \in P\}/F \\ &\cong M[\Pi\{B \restriction p : p \in P\}]/F \\ &\cong M[B]/F\end{aligned}$$

and

$$\begin{aligned}&\vee \{p \in P : M_p[B \restriction p] \models \varphi(f_1(p), \dots, f_n(p))\} \\ &= \vee \{p \in P : M[B \restriction p] \models \varphi(f_1(p), \dots, f_n(p))\} \\ &= \vee \{p \in P : M \models \varphi(f_1(p)(q), \dots, f_n(p)(q)) \text{ for all } q \in Q_p, \\ &\quad \text{a common domain of } f_1(p), \dots, f_n(p)\} \\ &= \vee \{q \in Q_p, p \in P : M \models \varphi(f_1(p)(q), \dots, f_n(p)(q))\}.\end{aligned}$$

(1) Hence Los's Theorem for  $(B, P)$ -ultraproducts coincides with the version for Boolean ultrapowers if the models  $M_p, p \in P$  are identical.

(b) If  $B \restriction p = 2$  for all  $p \in P$ , then

$$\begin{aligned}\Pi\{M_p[B \restriction p] : p \in P\}/F &= \Pi\{M_p[2] : p \in P\}/F \\ &\cong \Pi\{M_p : p \in P\}/F\end{aligned}$$

and

$$\begin{aligned}&\vee \{p \in P : M_p[B \restriction p] \models \varphi(f_1(p), \dots, f_n(p))\} \\ &= \vee \{p \in P : M_p[2] \models \varphi(f_1(p), \dots, f_n(p))\} \\ &= \vee \{p \in P : M_p \models \varphi(f_1(p)(1), \dots, f_n(p)(1))\}.\end{aligned}$$

And hence, under the complete Boolean algebra embedding  $i_P : \mathcal{P}(P) \rightarrow B$  defined by

$$i_P(U) = \vee U,$$

Los's Theorem for  $(B, P)$ -ultraproducts also coincides with the ordinary version if  $B \restriction p = 2$  for all  $p \in P$ .

---

## Chapter 1

### Boolean ultrapowers which are not isomorphic to ordinary ultrapowers

The fact that Boolean ultrapowers are a natural generalisation of ordinary ultrapowers poses the question whether every Boolean ultrapower is isomorphic to an ordinary ultrapower. Mansfield [1971] alludes to this question in his paper and provides a near example using the class of ordinals,  $O_n$ , which answers this question in the negative. B. Koppelberg and S. Koppelberg [1976] finally settle this question with their construction of a Boolean ultrapower which is not isomorphic to any ordinary ultrapower. However, the cardinality of the language and the model are rather large. B. Koppelberg [1980] remarks that W. Lange has shown that there exist examples of such Boolean ultrapowers of which the language has cardinality  $\aleph_1$  and the model has cardinality  $2^{\aleph_1}$ . B. Koppelberg [1980] subsequently constructs similar examples of which the language has at most one binary relation symbol and the model has cardinality  $\aleph_0$  or  $\aleph_1$  under very strong set-theoretical assumptions. Such constructions naturally lead to the following problem: Is it possible to construct such Boolean ultrapowers of which the language is countable within ZFC?

In this chapter the constructions of B. Koppelberg [1976, 1980] and S. Koppelberg [1976] of Boolean ultrapowers which are not isomorphic to ordinary ultrapowers are presented in detail. The definition of the Boolean power as given by Foster [1953] will be used in the proofs.

#### 1.1 The original construction of B. Koppelberg and S. Koppelberg [1976] of a Boolean ultrapower which is not isomorphic to any ordinary ultrapower

The following relation between subalgebras of a Boolean algebra introduces the discussion.

##### Definition 1.1.1 :

Suppose that  $A$  and  $C$  are subalgebras of a Boolean algebra  $B$ .  $A$  and  $C$  are *independent* if  $a \in A$ ,  $c \in C$  and  $a, c > 0$  implies that  $a \wedge c > 0$ .

The *cofinality* of a partially ordered set  $\langle X, < \rangle$ , written  $cf X$ , is the least cardinal  $\alpha$  such that  $\langle X, < \rangle$  has a cofinal subset of cardinality  $\alpha$ . The independence relation between subalgebras of a Boolean algebra ensures the existence of an ultrafilter  $D$  on some complete Boolean algebra  $B$  such that  $\langle \omega, < \rangle[B]/D$  has large cofinality, and hence large cardinality.

**Lemma 1.1.2 [B. Koppelberg and S. Koppelberg 1976] :**

*Suppose that  $A$  and  $C$  are complete subalgebras of a complete Boolean algebra  $B$ . Suppose that  $C$  is infinite and that  $A$  and  $C$  are independent. Then if  $D$  is an ultrafilter on  $A$ , there exist an ultrafilter  $F$  on  $B$  and some  $g \in \langle \omega, < \rangle[C]$  such that  $F \cap A = D$  and*

$$\langle \omega, < \rangle[B]/F \models f/F < g/F$$

*for each  $f \in \langle \omega, < \rangle[A]$ . (Note that  $\langle \omega, < \rangle[A]/F \cap A$  is elementarily embeddable into  $\langle \omega, < \rangle[B]/F$  since  $A$  is a complete subalgebra of  $B$ .)*

**Proof :**

Suppose that  $g \in \langle \omega, < \rangle[C]$  is such that  $g(n) > 0$  for each  $n \in \omega$ , and  $F$  is an ultrafilter on  $B$  such that

$$D \cup \{\|f < g\|_{\langle \omega, < \rangle[B]} : f \in \langle \omega, < \rangle[A]\} \subseteq F.$$

It suffices to show that  $D \cup \{\|f < g\|_{\langle \omega, < \rangle[B]} : f \in \langle \omega, < \rangle[A]\}$  has the finite intersection property, which would imply the existence of  $F$ . Suppose not. Then there exist  $d \in D$  and some  $f \in \langle \omega, < \rangle[A]$  such that  $d \wedge \|f < g\|_{\langle \omega, < \rangle[B]} = 0$ . Hence

$$\begin{aligned} d \wedge \|g \leq f\|_{\langle \omega, < \rangle[B]} &= (d \wedge \|g \leq f\|_{\langle \omega, < \rangle[B]}) \vee (d \wedge \|f < g\|_{\langle \omega, < \rangle[B]}) \\ &= d \wedge (\|g \leq f\|_{\langle \omega, < \rangle[B]} \vee \|f < g\|_{\langle \omega, < \rangle[B]}) \\ &= d \end{aligned}$$

$$\text{i.e. } d \leq \|g \leq f\|_{\langle \omega, < \rangle[B]} = \vee \{g(m) \wedge f(n) : m, n \in \omega \text{ and } m \leq n\}.$$

Since

$$0 < d = d \wedge \vee \{f(n) : n \in \omega\} = \vee \{d \wedge f(n) : n \in \omega\},$$

there exists  $\tilde{n} \in \omega$  such that  $d \wedge f(\tilde{n}) > 0$ . The independence of  $A$  and  $C$  implies that  $d \wedge f(\tilde{n}) \wedge g(\tilde{n} + 1) > 0$ . Furthermore,

$$d \wedge f(\tilde{n}) \wedge g(\tilde{n} + 1) \leq d \leq \vee \{g(m) \wedge f(n) : m, n \in \omega \text{ and } m \leq n\}$$

i.e.  $\vee \{d \wedge f(\tilde{n}) \wedge g(\tilde{n} + 1) \wedge g(m) \wedge f(n) : m, n \in \omega \text{ and } m \leq n\} > 0$ , and since  $f(\tilde{n}) \wedge f(n) = 0$  if  $n \neq \tilde{n}$ , it follows that

$$d \wedge f(\tilde{n}) \wedge g(\tilde{n} + 1) \leq \vee \{g(m) \wedge f(\tilde{n}) : m \in \omega \text{ and } m \leq \tilde{n}\}.$$

But if  $m \leq \tilde{n}$ , then  $m \neq \tilde{n} + 1$ , and thus  $g(m) \wedge g(\tilde{n} + 1) = 0$ . Then

$$\begin{aligned} 0 < d \wedge f(\tilde{n}) \wedge g(\tilde{n} + 1) &= \vee \{d \wedge f(\tilde{n}) \wedge g(\tilde{n} + 1) \wedge g(m) \wedge f(\tilde{n}) : m \in \omega, m \leq \tilde{n}\} \\ &= 0, \quad \text{a contradiction.} \end{aligned}$$

□

The following properties of ultrafilters need to be noted in order to derive a lemma on ultrafilters on powerset Boolean algebras.

**Definition 1.1.3 :**

Suppose that  $\delta$  is an infinite cardinal,  $I$  is a set and  $F$  is an ultrafilter on  $\mathcal{P}(I)$ .  $F$  is  $\delta$ -decomposable if there exists a partition  $\{I_\alpha : \alpha < \delta\}$  of  $I$  such that for each  $U \subseteq \delta$  with  $|U| < \delta$ ,  $\cup\{I_\alpha : \alpha \in U\} \notin F$ . Otherwise,  $F$  is  $\delta$ -indecomposable.

**Definition 1.1.4 :**

Suppose that  $F$  and  $G$  are ultrafilters on  $\mathcal{P}(I)$  and  $\mathcal{P}(J)$  respectively. Then  $F \leq G$  in the *Rudin-Keisler ordering* if there exists a function  $f : J \rightarrow I$  such that

$$\text{for each } X \subseteq I, X \in F \text{ iff } f^{-1}[X] \in G.$$

**Lemma 1.1.5 [B. Koppelberg and S. Koppelberg 1976] :**

Suppose that  $\lambda$  and  $\mu$  are infinite cardinals, and  $F$  is an ultrafilter on  $\mathcal{P}(I)$  which is  $\delta$ -indecomposable for all  $\mu < \delta \leq 2^{(\lambda^\mu)^+}$ . Then there exist  $\tilde{\mu} \leq \mu$  and an ultrafilter  $G$  on  $\mathcal{P}(\tilde{\mu})$  such that  $\langle \lambda, < \rangle^I / F \cong \langle \lambda, < \rangle^{\tilde{\mu}} / G$ . Moreover,  $G \leq F$  in the *Rudin-Keisler ordering*.

**Proof :**

The fact that  $F$  is  $\delta$ -indecomposable for all  $\mu < \delta \leq 2^{(\lambda^\mu)^+}$  implies that for each function  $f$  on  $I$  of which the range has cardinality at most  $2^{(\lambda^\mu)^+}$ , there exists  $J \subseteq I$  such that  $J \in F$  and the cardinality of the range of  $f \upharpoonright J$  is at most  $\mu$ . Suppose that  $|\langle \lambda, < \rangle^I / F| = \kappa$ . By the above remark, there exists a sequence  $\langle f_\alpha : \alpha < \kappa \rangle$  in  $\lambda^I$  such that  $\lambda^I / F = \{f_\alpha / F : \alpha < \kappa\}$ ,  $\alpha < \beta < \kappa$  implies that  $f_\alpha / F \neq f_\beta / F$ , and the range of  $f_\alpha$  has cardinality at most  $\mu$ . Then  $\kappa \leq \lambda^\mu$ . Suppose not. For each  $\alpha < (\lambda^\mu)^+$ , suppose that  $P_\alpha$  is the partition of  $I$  induced by  $f_\alpha : I \rightarrow \lambda$ , and  $P$  is the coarsest common refinement of the  $P_\alpha$ ,  $\alpha < (\lambda^\mu)^+$ . Then, since  $|P_\alpha| \leq \mu$  for each  $\alpha$ ,

$$|P| \leq \mu^{(\lambda^\mu)^+} = 2^{(\lambda^\mu)^+},$$

and suppose that  $P = \{I_\nu : \nu < 2^{(\lambda^\mu)^+}\}$ . By the  $\delta$ -indecomposability of  $F$ , there exists  $U \subseteq 2^{(\lambda^\mu)^+}$  such that  $\cup\{I_\nu : \nu \in U\} \in F$  and  $|U| \leq \mu$ . If  $\nu < 2^{(\lambda^\mu)^+}$  and  $\alpha < (\lambda^\mu)^+$ , then  $f_\alpha \upharpoonright I_\nu$  is a constant function. Hence

$$|\{f_\alpha : \alpha < (\lambda^\mu)^+\}| \leq \lambda^\mu, \text{ a contradiction,}$$

and  $\kappa \leq \lambda^\mu$ .

Now suppose that  $P_\alpha, \alpha < \kappa$  and  $P$  are defined as above. Since  $|P| \leq \mu^{(\lambda^\mu)} = 2^{(\lambda^\mu)}$ , there exist  $\tilde{\mu} \leq \mu$  and an injective sequence  $\langle I_\nu : \nu < \tilde{\mu} \rangle$  in  $P$  such that  $\cup\{I_\nu : \nu < \tilde{\mu}\} \in F$ . Define a function  $g : I \rightarrow \tilde{\mu}$  by

$$g(i) = \begin{cases} \nu, & i \in I_\nu \\ 0, & i \in I - \cup\{I_\nu : \nu < \tilde{\mu}\} \end{cases}$$

and an ultrafilter  $G$  on  $\mathcal{P}(\tilde{\mu})$  by

$$\text{for each } X \subseteq \tilde{\mu}, X \in G \text{ iff } g^{-1}[X] \in F.$$

If  $g_\alpha(\nu)$  is the constant value of  $f_\alpha \upharpoonright I_\nu$  for  $\nu < \tilde{\mu}$ , then the function  $H : \langle \lambda, < \rangle^I / F \rightarrow \langle \lambda, < \rangle^{\tilde{\mu}} / G$  defined by

$$H(f_\alpha / F) = g_\alpha / G, \quad \alpha < \kappa,$$

is an isomorphism. □

A Boolean algebra  $B$  satisfies the *countable chain condition* if every pairwise disjoint subset of  $B$  is at most countable. If  $B$  is a free Boolean algebra and  $X \subseteq B$  such that  $|X| > \aleph_0$ , then there exists  $Y \subseteq X$  such that  $|Y| > \aleph_0$  and  $Y$  is independent. Thus every free Boolean algebra  $B$  satisfies the countable chain condition.

Now suppose that  $\kappa$  is a cardinal such that  $\kappa^{\aleph_0} = \kappa$  and  $\kappa > 2^{\aleph_0}$ . (An example of such a cardinal is  $2^{2^{\aleph_0}}$ .) The following lemma describes the construction of an ultrafilter  $D$  on a complete Boolean algebra  $B$  such that  $cf\langle \omega, < \rangle[B]/D = cf\kappa$  and  $|\langle \omega, < \rangle[B]/D| = \kappa$ .

**Lemma 1.1.6 [B. Koppelberg and S. Koppelberg 1976] :**

*Suppose that  $Fr_\kappa$  is the free Boolean algebra on  $\kappa$  generators and  $B$  is the completion of  $Fr_\kappa$ . Then there exists an ultrafilter  $D$  on  $B$  such that  $cf\langle \omega, < \rangle[B]/D = cf\kappa$  and  $|\langle \omega, < \rangle[B]/D| = \kappa$ .*

**Proof :**

Suppose that  $V$  is a set of free generators for  $Fr_\kappa$  (hence  $|V| = \kappa$ ), and  $\{V_\alpha : \alpha < \kappa\}$  is a partition of  $V$  such that  $|V_\alpha| \geq \aleph_0$  for each  $\alpha < \kappa$ . Suppose that  $B_\alpha$  is the subalgebra of  $B$  completely generated by  $\cup\{V_\gamma : \gamma < \alpha\}$  and write  $B_\alpha = \overline{[\cup\{V_\gamma : \gamma < \alpha\}]}$ . Now  $B_{\alpha+1} = \overline{[B_\alpha \cup \overline{V_\alpha}]}$ , and  $B_\alpha$  and  $\overline{V_\alpha}$  are independent subalgebras of  $B_{\alpha+1}$ . By Lemma 1.1.2, define ultrafilters  $D_\alpha$  on  $B_\alpha$  by induction such that  $D_\beta \cap B_\alpha = D_\alpha$  for  $\alpha < \beta < \kappa$ , and for each  $\alpha < \kappa$  there exists  $g_{\alpha+1} \in \langle \omega, < \rangle[B_{\alpha+1}]$  satisfying

$$\langle \omega, < \rangle[B_{\alpha+1}]/D_{\alpha+1} \models f/D_{\alpha+1} < g_{\alpha+1}/D_{\alpha+1}$$

for each  $f \in \langle \omega, < \rangle[B_\alpha]$ .

Now  $cf\kappa = cf\aleph_0 > \aleph_0$ . Also,  $Fr_\kappa$  (and hence  $B$ ) satisfies the countable chain condition. Thus  $B = \cup\{B_\alpha : \alpha < \kappa\}$  and  $\langle \omega, < \rangle[B] = \cup\{\langle \omega, < \rangle[B_\alpha] : \alpha < \kappa\}$ .  $D = \cup\{D_\alpha : \alpha < \kappa\}$  is the union of an updirected family of ultrafilters on subalgebras of  $B$  and hence  $D$  is an ultrafilter on  $B$ . The set  $\{g_{\alpha+1}/D : \alpha < \kappa\}$  has order type  $\kappa$  and is cofinal in  $\langle \omega, < \rangle[B]/D$ , and thus  $cf\langle \omega, < \rangle[B]/D = cf\kappa$ . Finally,  $|B| = \kappa^{\aleph_0} = \kappa$ , and  $|\langle \omega, < \rangle[B]| \leq |B^\omega| \leq \kappa^{\aleph_0} = \kappa$ , which implies that  $|\langle \omega, < \rangle[B]/D| = \kappa$ .  $\square$

The groundwork for the Koppelbergs' [1976] construction of a Boolean ultrapower which is not isomorphic to any ordinary ultrapower is now complete. The object of Lemma 1.1.5 and Lemma 1.1.6 is to show that if this Boolean ultrapower were isomorphic to any ordinary ultrapower, then they would have isomorphic submodels which have different cardinalities. The details of the construction are given in the following theorem.

**Theorem 1.1.7 [B. Koppelberg and S. Koppelberg 1976] :**

Suppose that  $\tau = 2^{(2^{\aleph_0})^+}$ , and  $L = \{<\} \cup \{\bar{R}_\alpha : \alpha < \tau^{\aleph_0}\}$  where  $\bar{R}_\alpha$  is a unary relation symbol for each  $\alpha < \tau^{\aleph_0}$ . Define a model  $M$  by

$$M = \langle \tau, <, \{R_\alpha : \alpha < \tau^{\aleph_0}\} \rangle$$

where  $<$  is the usual ordering on  $\tau$  and

$$\{R_\alpha : \alpha < \tau^{\aleph_0}\} = \{U \subseteq \tau : |U| \leq \aleph_0\}.$$

If  $B$  and  $D$  are as in Lemma 1.1.6, then  $M[B]/D$  is not isomorphic to any ultrapower of  $M$ .



**Proof :**

Suppose not. Then there exist a set  $I$  and an ultrafilter  $F$  on  $\mathcal{P}(I)$  such that  $H : M[B]/D \rightarrow M^I/F$  is an isomorphism. For  $g \in M^I$ , suppose that  $f \in M[B]$  such that

$$H(f/D) = g/F.$$

Since  $B$  satisfies the countable chain condition, there exists  $\alpha < \tau^{\aleph_0}$  such that

$$\{m \in M : f(m) > 0\} \subseteq R_\alpha.$$

Thus

$$M[B]/D \models \bar{R}_\alpha(f/D) \text{ and } M^I/F \models \bar{R}_\alpha(g/F),$$

which implies that there exists some  $J \subseteq I$  such that  $J \in F$  and the range of  $g \upharpoonright J$  has cardinality at most  $\aleph_0$ . Hence  $F$  is  $\delta$ -indecomposable for all  $\aleph_0 < \delta \leq 2^{(2^{\aleph_0})^+} = \tau$ . By Lemma 1.1.5, there exist  $\tilde{\mu} \leq \aleph_0$  and an ultrafilter  $G$  on  $\mathcal{P}(\tilde{\mu})$  such that  $\langle \omega, < \rangle^I/F \cong \langle \omega, < \rangle^{\tilde{\mu}}/G$ , and hence  $|\langle \omega, < \rangle^I/F| = |\langle \omega, < \rangle^{\tilde{\mu}}/G| \leq 2^{\aleph_0}$ . By Lemma 1.1.6 however,  $|\langle \omega, < \rangle[B]/D| = \kappa > 2^{\aleph_0}$ , contradicting the assumption that  $H$  is an isomorphism between  $M[B]/D$  and  $M^I/F$ , since it would induce an isomorphism between  $\langle \omega, < \rangle[B]/D$  and  $\langle \omega, < \rangle^I/F$ .  $\square$

In the above theorem the cardinality of the model (i.e.  $2^{(2^{\aleph_0})^+}$ ) and that of the language (i.e.  $(2^{(2^{\aleph_0})^+})^{\aleph_0}$ ) is quite large, and B. Koppelberg [1980] thus constructs examples for which the language has at most one binary relation symbol and the model has cardinality  $\aleph_0$  or  $\aleph_1$ . In what follows, full details of such constructions are given. Note that Koppelberg uses Foster's [1953] definition of a Boolean power for these proofs as well.

The following property pertaining to ultrafilters is relevant to Koppelberg's constructions. It is sufficient to consider these ultrafilters on powersets of cardinals rather than on arbitrary powersets.

## 1.2 The constructions of B. Koppelberg [1980] of Boolean ultrapowers of $\langle \omega, < \rangle$ and $\langle \omega_1, < \rangle$ which are not isomorphic to any ordinary ultrapowers

### Definition 1.2.1 :

Suppose that  $\alpha, \tau$  and  $\lambda$  are infinite cardinals, and  $F$  is an ultrafilter on  $\mathcal{P}(\alpha)$ .  $F$  is  $(\tau, \lambda)$ -regular if there exists  $E \subseteq F$  such that  $|E| = \lambda$  and for each  $\tilde{E} \subseteq E$  with  $|\tilde{E}| \geq \tau$ ,  $\cap\{X : X \in \tilde{E}\} = \emptyset$ .

For the first of these constructions, Koppelberg [1980] devises the following lemma.

**Lemma 1.2.2 [B. Koppelberg 1980] :**

*Suppose that  $\tau$  is regular and  $F$  is  $(\tau, \lambda)$ -regular on  $\mathcal{P}(\alpha)$ . Then  $cf\langle\tau, <\rangle^\alpha / F > \lambda$ .*

**Proof :**

Since  $F$  is  $(\tau, \lambda)$ -regular, there exists  $E \subseteq F$  such that  $|E| = \lambda$  and for each  $\tilde{E} \subseteq E$  with  $|\tilde{E}| \geq \tau$ ,  $\cap\{X : X \in \tilde{E}\} = \emptyset$ . Suppose that  $H \subseteq \tau^\alpha$  and  $|H| = \lambda$ . Now suppose that  $\langle e_\xi : \xi < \lambda \rangle$  and  $\langle h_\xi : \xi < \lambda \rangle$  are enumerations of  $E$  and  $H$  respectively such that  $\xi < \eta < \tau$  implies that  $e_\xi \neq e_\eta$ . Define  $f \in \tau^\alpha$  by

$$h_\xi(i) < f(i), \quad i \in e_\xi.$$

This is possible since  $|\{\xi < \lambda : i \in e_\xi\}| < \tau$  and  $\tau$  is regular. Thus,  $e_\xi \subseteq \{i \in \alpha : h_\xi(i) < f(i)\}$  and hence  $h_\xi/F < f/F$  for each  $\xi < \lambda$ .  $\square$

In addition, Koppelberg requires the following result of Kunen and Prikry [1971].

**Theorem 1.2.3 [K. Kunen and K. Prikry 1971] :**

*Suppose that  $\delta$  is regular and  $F$  is  $\delta^+$ -decomposable on  $\mathcal{P}(\alpha)$ . Then  $F$  is  $\delta$ -decomposable.*

An ultrafilter  $F$  on  $\mathcal{P}(\alpha)$  is *uniform* if for each  $X \in F$ ,  $|X| = \alpha$ .

**Remark 1.2.4 :**

An ultrafilter  $F$  on  $\mathcal{P}(\alpha)$  is  $\delta$ -decomposable iff there exists a uniform ultrafilter  $G$  on  $\mathcal{P}(\delta)$  such that  $G \leq F$  in the Rudin-Keisler ordering.

The first construction now follows.

**Theorem 1.2.5 [B. Koppelberg 1980] :**

*Suppose that  $2^{(2^{\aleph_0})^+} < \aleph_\omega$ , and  $\langle\omega_1, <\rangle[B]/D$  is a Boolean ultrapower such that  $B$  satisfies the countable chain condition and  $|\langle\omega_1, <\rangle[B]/D| > 2^{\aleph_0}$ . Then  $\langle\omega_1, <\rangle[B]/D$  is not isomorphic to any ultrapower of  $\langle\omega_1, <\rangle$ .*

**Proof :**

Suppose that  $\alpha$  is an infinite cardinal and  $F$  is an ultrafilter on  $\mathcal{P}(\alpha)$  such that  $\langle\omega_1, <\rangle[B]/D \cong \langle\omega_1, <\rangle^\alpha / F$ . If  $F$  is  $\aleph_n$ -decomposable for some  $n \in \omega - \{0\}$ ,

then  $F$  is  $\aleph_1$ -decomposable by Theorem 1.2.3. By Remark 1.2.4, there exists a uniform ultrafilter  $G$  on  $\mathcal{P}(\aleph_1)$  such that  $G \leq F$  in the Rudin-Keisler ordering. Now  $G$ , and hence  $F$  is  $(\aleph_1, \aleph_1)$ -regular, and by Lemma 1.2.2, it follows that  $cf\langle\omega_1, <\rangle^\alpha / F > \aleph_1$ . However, since  $B$  satisfies the countable chain condition,  $cf\langle\omega_1, <\rangle[B]/D = \aleph_1$ , contradicting the assumption that  $\langle\omega_1, <\rangle[B]/D \cong \langle\omega_1, <\rangle^\alpha / F$ . On the other hand, if  $F$  is  $\delta$ -indecomposable for all  $\delta$  satisfying  $\aleph_0 < \delta \leq 2^{(2^{\aleph_0})^+} < \aleph_\omega$ , then by Lemma 1.1.5, there exists an ultrafilter  $G$  on  $\mathcal{P}(\aleph_0)$  such that  $\langle\omega, <\rangle^\alpha / F \cong \langle\omega, <\rangle^{\aleph_0} / G$ . Then  $|\langle\omega, <\rangle^\alpha / F| \leq 2^{\aleph_0}$ , but  $|\langle\omega, <\rangle[B]/D| > 2^{\aleph_0}$ , again contradicting the assumption that  $\langle\omega_1, <\rangle[B]/D \cong \langle\omega_1, <\rangle^\alpha / F$ , since  $\langle\omega, <\rangle[B]/D \cong \langle\omega, <\rangle^\alpha / F$  under the induced isomorphism.  $\square$

For the second construction, the notion of  $0^\sharp$  (zero-sharp) described below is required. (See Jech [1972].)  $0^\sharp$  was first introduced by J. Silver in 1971. Its existence is linked with the structure of the (real) universe relative to the constructible universe. It is in fact a “large cardinal axiom”. The second construction is based on the assumption that  $0^\sharp$  does not exist.

**Definition 1.2.6 :**

A set  $X \subseteq Y$  is *definable* over  $Y$  if there exist a formula  $\varphi$  in the language of set theory and  $y_1, \dots, y_n \in Y$  such that

$$X = \{x \in Y : \langle Y, \in \rangle \models \varphi(x, y_1, \dots, y_n)\}.$$

$\text{Def}(Y)$  is the set of all subsets definable over  $Y$ .

The class  $\mathcal{L}$  of *constructible sets* is defined as follows:

$$\begin{aligned} L_0 &= \emptyset \\ L_\alpha &= \cup\{L_\gamma : \gamma < \alpha\} \text{ if } \alpha \text{ is a limit ordinal} \\ L_{\alpha+1} &= \text{def}(L_\alpha) \\ \mathcal{L} &= \cup\{L_\alpha : \alpha \in O_n, \text{ the class of all ordinals}\} \end{aligned}$$

Define  $0^\sharp = \{\varphi : L_{\aleph_\omega} \models \varphi(\aleph_1, \dots, \aleph_n)\}$ .

Koppelberg also requires the following theorems for his construction.

**Theorem 1.2.7 [Ketonen 1976] :**

*Suppose that  $\alpha$  is a successor cardinal and  $F$  is uniform on  $\mathcal{P}(\alpha)$ . Then there exists  $\tau < \alpha$  such that  $F$  is  $(\tau, \alpha)$ -regular.*

**Theorem 1.2.8 [B. Koppelberg]:**

*Suppose that  $0^\sharp$  does not exist. Suppose that  $\alpha$  is regular and  $F$  is uniform on  $\mathcal{P}(\alpha)$ . Then  $F$  is  $\delta$ -decomposable for each regular  $\delta \leq \alpha$ .*

**Theorem 1.2.9 [Jensen] :**

Suppose that  $0^\#$  does not exist. Suppose that  $\kappa$  is an infinite singular cardinal. Then  $2^{<\kappa} = \lambda$  if  $2^\gamma = \lambda$  for some  $\gamma < \kappa$ ; otherwise  $2^\kappa = \lambda^+$ .

The second construction now follows.

**Theorem 1.2.10 [B. Koppelberg 1980] :**

Suppose that  $0^\#$  does not exist. Suppose that  $\langle \omega, < \rangle[B]/D$  is a Boolean ultrapower such that  $|\langle \omega, < \rangle[B]/D| = \kappa$ ,  $cf\langle \omega, < \rangle[B]/D = \aleph_1$  where  $\kappa$  is a strong limit cardinal such that either  $\kappa > cf\kappa > \aleph_0$ , or  $2^\kappa = \kappa^{(n)}$  for some  $n \in \omega$ . Then  $\langle \omega, < \rangle[B]/D$  is not isomorphic to any ultrapower of  $\langle \omega, < \rangle$ .

**Proof :**

By the assumptions on  $\kappa$ ,  $\kappa^{\aleph_0} = \kappa$ . Suppose that  $F$  is an ultrafilter on  $\mathcal{P}(\alpha)$  such that  $\langle \omega, < \rangle[B]/D \cong \langle \omega, < \rangle^\alpha/F$ . W.l.o.g., it can be assumed that  $F$  is uniform on  $\mathcal{P}(\alpha)$ . Now suppose that  $F$  is  $\delta$ -decomposable for some  $\delta$  satisfying  $cf\delta > \aleph_0$ . Then  $F$  is  $cf\delta$ -decomposable, and by Remark 1.2.4, there exists a uniform ultrafilter  $G$  on  $\mathcal{P}(cf\delta)$  such that  $G \leq F$  in the Rudin-Keisler ordering. Similarly, since  $G$  is  $\aleph_1$ -decomposable by Theorem 1.2.8, there exists a uniform ultrafilter  $E$  on  $\mathcal{P}(\aleph_1)$  such that  $E \leq G$  in the Rudin-Keisler ordering. By Theorem 1.2.7,  $E$  and hence  $F$ , is  $(\aleph_0, \aleph_1)$ -regular, and by Lemma 1.2.2,  $cf\langle \omega, < \rangle^\alpha/F > \aleph_1$ , contradicting the assumption that  $\langle \omega, < \rangle[B]/D \cong \langle \omega, < \rangle^\alpha/F$ , since  $cf\langle \omega, < \rangle[B]/D = \aleph_1$ . Otherwise, if  $F$  is  $\delta$ -decomposable, then  $cf\delta = \aleph_0$ . Since  $|\langle \omega, < \rangle[B]/D| = \kappa$ , there exists a sequence  $\langle f_\xi : \xi < \kappa \rangle$  in  $\omega^\alpha$  such that for  $\xi < \eta < \kappa$ ,  $f_\xi/F \neq f_\eta/F$ . Thus, for  $\xi < \kappa$  and  $n \in \omega$ ,  $\{i \in \alpha : f_\xi(i) = n\} \notin F$ . Suppose that  $P_\xi$  is the partition of  $\alpha$  induced by  $f_\xi$ . Now  $|P_\xi| \leq \aleph_0$  for each  $\xi < \kappa$ , and if  $P$  is the coarsest common refinement of the  $P_\xi, \xi < \kappa$ , then  $|P| \leq \aleph_0^\kappa = 2^\kappa$ . By Theorem 1.2.9,  $2^\kappa = \kappa^+$  or  $2^\kappa = \kappa^{(n)}$  for some  $n \in \omega$ . Suppose that  $\tau$  is the smallest cardinal such that there exists an injective sequence  $\langle I_\nu : \nu < \tau \rangle$  in  $P$  satisfying  $\cup\{I_\nu : \nu < \tau\} \in F$ . Hence  $\tau \geq \aleph_0$  and  $F$  is  $\tau$ -decomposable, which implies that  $cf\tau = \aleph_0$  and  $\tau \neq \kappa^{(n)}$  for any  $n \in \omega$  since  $cf\kappa > \aleph_0$ . Thus  $\tau < \kappa$ . If  $\xi < \kappa$  and  $\nu < \tau$ , then  $f_\xi \upharpoonright I_\nu$  is constant. Define a sequence  $\langle g_\xi : \xi < \kappa \rangle$  in  $\omega^\tau$  by

$$g_\xi(\nu) = \text{the constant value of } f_\xi \upharpoonright I_\nu.$$

For  $\xi < \eta < \kappa$ ,  $\cup\{I_\nu : \nu < \tau\} \cap \{i \in \alpha : f_\xi(i) \neq f_\eta(i)\} \neq \emptyset$  (since  $\cup\{I_\nu : \nu < \tau\} \cap \{i \in \alpha : f_\xi(i) \neq f_\eta(i)\} \in F$ ), and thus  $g_\xi(\nu) \neq g_\eta(\nu)$ . Then  $g_\xi \neq g_\eta$  which implies that  $\kappa \leq \aleph_0^\tau = 2^\tau$ , contradicting the fact that  $\kappa$  is a strong limit cardinal and  $\tau < \kappa$ .  $\square$

For the third construction, the following lemma of Koppelberg [1980] is also required.

**Lemma 1.2.11 [B. Koppelberg 1980] :**

*Suppose that  $0^\sharp$  does not exist. If  $\alpha$  is regular,  $2^\alpha = \alpha^+$ , and  $F$  is uniform on  $\mathcal{P}(\alpha)$ , then  $F$  is  $(\aleph_0, \lambda)$ -regular for each  $\lambda < \alpha$ .*

Now suppose that  $LCH$  is the statement that all limit cardinals are strong limit cardinals. The third construction requires the assumption that  $LCH$  holds.

**Theorem 1.2.12 [B. Koppelberg 1980] :**

*Assume that  $0^\sharp$  does not exist and  $LCH$  holds. Then for each infinite cardinal  $\alpha$  and any ultrafilter  $F$  on  $\mathcal{P}(\alpha)$ ,  $|\langle \omega, < \rangle^\alpha / F|$  is regular. If  $\alpha$  is a singular cardinal, then either  $|\langle \omega, < \rangle^\alpha / F| < \alpha$  or  $|\langle \omega, < \rangle^\alpha / F| = \alpha^+$ .*

**Proof :**

Suppose that there exist an infinite cardinal  $\alpha$  and an ultrafilter  $F$  on  $\mathcal{P}(\alpha)$  such that  $|\langle \omega, < \rangle^\alpha / F| = \lambda$ , for some singular cardinal  $\lambda$ . By Exercise 4.3.15 in Chang and Keisler [1973],  $\lambda^{\aleph_0} = \lambda$  and hence  $\lambda > \aleph_0$ . Define  $\tau$  by

$$\tau = \vee \{ \delta : \delta \text{ is a regular cardinal and } F \text{ is } \delta\text{-decomposable} \}.$$

If  $\tau < \lambda$ , then by  $LCH$  and Theorem 1.2.3,  $F$  is  $\mu$ -indecomposable for  $\tau < \mu \leq 2^{(2^\tau)^+}$ . By Lemma 1.1.5, there exist  $\tilde{\alpha} \leq \tau < \lambda$  and an ultrafilter  $G$  on  $\mathcal{P}(\tilde{\alpha})$  such that  $G \leq F$  in the Rudin-Keisler ordering and  $\langle \omega, < \rangle^\alpha / F \cong \langle \omega, < \rangle^{\tilde{\alpha}} / G$ . This implies that  $|\langle \omega, < \rangle^\alpha / F| = |\langle \omega, < \rangle^{\tilde{\alpha}} / G| \leq 2^{\tilde{\alpha}} < \lambda$ , a contradiction. On the other hand, if  $\tau \geq \lambda$ , then by  $LCH$  and the fact that  $cf \lambda > \aleph_0$ , the set

$$X = \{ \gamma < \lambda : \gamma \text{ is a strong limit cardinal} \}$$

is cofinal in  $\lambda$ . By Theorem 1.2.8,  $F$  is  $\gamma^+$ -decomposable for all  $\gamma \in X$ . Thus, by Remark 1.2.4, there exists a uniform ultrafilter  $E$  on  $\mathcal{P}(\gamma^+)$  such that  $E \leq F$  in the Rudin-Keisler ordering. By  $LCH$  and Theorem 1.2.9,  $2^{<\gamma^+} = 2^\gamma = \gamma^+$  for  $\gamma \in X$ . By Lemma 1.2.11,  $E$  is  $(\aleph_0, \delta)$ -regular for each  $\delta \leq \gamma$ . Hence  $F$  is  $(\aleph_0, \delta)$ -regular for each  $\delta < \lambda$ . Now by Lemma 1.2.2,  $cf \langle \omega, < \rangle^\alpha / F > \delta$  for all  $\delta < \lambda$ , which implies that  $|\langle \omega, < \rangle^\alpha / F| \geq cf \langle \omega, < \rangle^\alpha / F \geq \lambda^+$ , a contradiction.

For the second part of the theorem, suppose that  $\alpha$  is a singular cardinal and  $F$  is an ultrafilter on  $\mathcal{P}(\alpha)$  such that  $|\langle \omega, < \rangle^\alpha / F| \geq \alpha$ . Then by the above result and Theorem 1.2.9,  $\alpha^+ \leq |\langle \omega, < \rangle^\alpha / F| = 2^\alpha = \alpha^+$ . In this case it can be shown that  $F$  is  $(\aleph_0, \alpha)$ -regular.  $\square$

The final construction of Koppelberg [1980] answers W. Lange's question concerning the existence of Boolean ultrapowers of  $\langle \omega, < \rangle$  which have cardinalities that are not the cardinalities of any ordinary ultrapowers of  $\langle \omega, < \rangle$ .

**Theorem 1.2.13 [B. Koppelberg 1980] :**

*Suppose that  $0^\#$  does not exist and  $LCH$  holds. Suppose that  $\kappa$  is a singular cardinal and  $cf\kappa > \aleph_0$ . Then there exist a complete Boolean algebra  $B$  and an ultrafilter  $D$  on  $B$  such that  $\kappa = |\langle \omega, < \rangle[B]/D|$  is not the cardinality of any ultrapower of  $\langle \omega, < \rangle$ .*

**Proof :**

Since  $\kappa > cf\kappa > \aleph_0$ ,  $\kappa^{\aleph_0} = \kappa$  by  $LCH$ . By Lemma 1.1.6, there exist a complete Boolean algebra  $B$  and an ultrafilter  $D$  on  $B$  such that  $|\langle \omega, < \rangle[B]/D| = \kappa$ . The result follows by Theorem 1.2.12.  $\square$

---

## Chapter 2

### Finitely iterated Boolean ultrapowers

An elementary result concerning ordinary ultrapowers is that a finitely iterated ultrapower is isomorphic to a single ultrapower. More precisely, if  $M$  is a model,  $I$  and  $J$  are sets and  $F$  and  $G$  are ultrafilters on  $\mathcal{P}(I)$  and  $\mathcal{P}(J)$  respectively, then there exists an ultrafilter  $F \otimes G$  on  $\mathcal{P}(I \times J)$  such that

$$(M^I/F)^J/G \cong M^{I \times J}/F \otimes G,$$

where

$$F \otimes G = \{X \subseteq I \times J : \{j \in J : \{i \in I : (i, j) \in X\} \in F\} \in G\}.$$

Note that in general  $(M^I/F)^J/G \not\cong (M^J/G)^I/F$ , and hence the order of iteration is significant. Further details of the above result can be found in Chang and Keisler [1973] and Bell and Slomson [1971].

Since the notion of a Boolean ultrapower is a generalisation of the notion of an ordinary ultrapower under the isomorphism  $M^I/F \cong M[\mathcal{P}(I)]/F$ , it is logical to investigate the conditions under which a finitely iterated Boolean ultrapower may be expressed as a single ultrapower.

In this chapter the approaches of Ouwehand and Rose [1998] and Mansfield [1971] to the problem of finitely iterated Boolean ultrapowers will be discussed extensively. Besides the obvious differences in the method of proof which stem directly from Mansfield's use of Foster's [1953] definition of a Boolean power and Ouwehand and Rose's approach to Boolean powers as direct limits of direct powers, the conditions for finite iteration are also somewhat different. Whereas Mansfield focuses on Boolean ultrapowers (where the Boolean algebra is complete), Ouwehand and Rose consider  $\kappa$ -bounded Boolean ultrapowers (where the Boolean algebra is  $\kappa$ -complete).

#### 2.1 The approach of R. Mansfield [1971] to the problem of finitely iterated Boolean ultrapowers

Suppose that  $A$  and  $B$  are complete Boolean algebras. A new algebra  $A[B]$  may be defined as follows: The universe of  $A[B]$  is the Boolean power  $A[B]$ , but with the two-valued equality relation i.e. for  $f, g \in A[B]$ ,  $\|f = g\| = 1$  iff  $f = g$ .

Using the  $\Sigma$ -notation as in Theorem 0.4.4, it follows by Remark 0.4.5 (a) that each element of  $A[B]$  can be written in the form  $\Sigma\{b_i \cdot a_i : i \in I\}$  where

$\langle b_i : i \in I \rangle$  is a pairwise disjoint sequence in  $B$  with  $\vee\{b_i : i \in I\} = 1$ , and  $\langle a_i : i \in I \rangle$  is a sequence in  $A$  allowing repetitions. With this notation the operations in  $A[B]$  are:

- (a)  $(\Sigma\{b_i.a_i : i \in I\}) \vee (\Sigma\{\tilde{b}_j.\tilde{a}_j : j \in J\}) = \Sigma\{\Sigma\{(b_i \wedge \tilde{b}_j).(a_i \vee \tilde{a}_j) : j \in J\} : i \in I\},$
- (b)  $(\Sigma\{b_i.a_i : i \in I\}) \wedge (\Sigma\{\tilde{b}_j.\tilde{a}_j : j \in J\}) = \Sigma\{\Sigma\{(b_i \wedge \tilde{b}_j).(a_i \wedge \tilde{a}_j) : j \in J\} : i \in I\},$
- (c)  $(\Sigma\{b_i.a_i : i \in I\})^c = \Sigma\{b_i.a_i^c : i \in I\}.$

Both  $A$  and  $B$  are embeddable into  $A[B]$  under the embeddings  $e_A : A \rightarrow A[B]$  defined by

$$e_A(a) = 1.a$$

and  $e_B : B \rightarrow A[B]$  defined by

$$e_B(b) = b.1 + b^c.0.$$

Now suppose that  $F$  and  $G$  are ultrafilters on  $A$  and  $B$  respectively. Define  $F \otimes G$  on  $A[B]$  by

$$F \otimes G = \{\Sigma\{b_i.a_i : i \in I\} : \vee\{b_i : a_i \in F\} \in G\}.$$

The following proposition can easily be checked.

**Proposition 2.1.1 :**

$F \otimes G$  is an ultrafilter on  $A[B]$ .

**Remark 2.1.2 :**

An ultrafilter  $U$  on  $A[B]$  is of the form  $F \otimes G$  if  $\Sigma\{b_i.a_i : i \in I\} \in U$  implies that there exists  $i \in I$  such that  $a_i \in U$ .

The following proposition shows that the joins of sets of the form  $\{\Sigma\{b_i.a_{ij} : i \in I\} : j \in J\}$  always exist in  $A[B]$ , even though  $A[B]$  is not necessarily complete.

**Proposition 2.1.3 :**

$$\vee\{\Sigma\{b_i.a_{ij} : i \in I\} : j \in J\} = \Sigma\{b_i.\vee\{a_{ij} : j \in J\} : i \in I\}.$$

A Boolean algebra  $B$  satisfies the  $(\kappa, \lambda)$ -*distributive law* if for any sets  $I$  and  $J$  such that  $|I| \leq \kappa$  and  $|J| \leq \lambda$ , and for any family  $\{b_{ij} : i \in I, j \in J\}$  in  $B$ ,

$$\wedge\{\vee\{b_{ij} : j \in J\} : i \in I\} = \vee\{\wedge\{b_{if(i)} : i \in I\} : f \in J^I\},$$



provided that  $\bigvee\{b_{ij} : j \in J\}$  for  $i \in I$ ,  $\bigwedge\{\bigvee\{b_{ij} : j \in J\} : i \in I\}$  and  $\bigwedge\{b_{if(i)} : i \in I\}$  for  $f \in J^I$  exist in  $B$ . (See Monk and Bonnet [1989].)  $B$  satisfies the  $(\kappa, \infty)$ -distributive law if it satisfies the  $(\kappa, \lambda)$ -distributive law for all cardinals  $\lambda$ .

Mansfield requires the following well-known lemma to prove his main result.

**Lemma 2.1.4 :**

*Suppose that  $B$  is a complete Boolean algebra and  $B$  satisfies the  $(\kappa, \infty)$ -distributive law. If  $X$  is a set of partitions of  $B$  such that  $|X| < \kappa$ , then the partitions in  $X$  have a common refinement.*

**Proof :**

For each  $P \in X$ ,  $\bigvee\{p : p \in P\} = 1$ . Thus

$$\begin{aligned} 1 &= \bigwedge\{\bigvee\{p : p \in P\} : P \in X\} \\ &= \bigvee\{\bigwedge\{f(P) : P \in X\} : f \in B^X \text{ and } f(P) \in P\} \\ &\text{since } B \text{ satisfies the } (\kappa, \infty)\text{-distributive law.} \end{aligned}$$

Define

$$b_f = \bigwedge\{f(P) : P \in X\}$$

for each  $f$  such that  $f(P) \in P$ . Then  $\{b_f : f \in B^X \text{ and } f(P) \in P\}$  is a common refinement of the partitions in  $X$ .  $\square$

Note that the converse of Lemma 2.1.4 also holds. The reader is referred to Monk and Bonnet [1989], vol. 1, Proposition 14.9(c) for details of the proof.

**Definition 2.1.5 [R. Mansfield 1971] :**

Suppose that  $A$  and  $B$  are complete Boolean algebras.  $A$  satisfies the  $\kappa$ -chain condition w.r.t.  $B$  if for each function  $\tilde{H} : A \rightarrow B$  such that

$$\tilde{H}(a_1) \wedge \tilde{H}(a_2) > 0 \text{ implies that } a_1 = a_2 \text{ or } a_1 \wedge a_2 = 0,$$

there exists a set  $\{b_i : i \in I\}$  in  $B$  with  $\bigvee\{b_i : i \in I\} = 1$  and for each  $i$ ,

$$|\{a : \tilde{H}(a) \wedge b_i > 0\}| < \kappa.$$

The main result of Mansfield [1971] now follows.

**Theorem 2.1.6 [R. Mansfield 1971] :**

Suppose that  $A$  and  $B$  are complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $A$  and  $B$  respectively. If  $B$  satisfies the  $(\kappa, \infty)$ -distributive law,  $A$  satisfies the  $\kappa$ -chain condition w.r.t.  $B$  and  $A[B]$  is complete, then

$$M[A[B]]/F \otimes G \cong (M[A]/F)[B]/G.$$

**Proof :**

Suppose that  $\{b_i : i \in I\}$  is a pairwise disjoint set in  $B$  with  $\bigvee \{b_i : i \in I\} = 1$ , and  $\{f_i : i \in I\}$  is an arbitrary set in  $M[A]$ . Suppose that  $\Sigma\{b_i.f_i : i \in I\}$  represents the function from  $M$  into  $A[B]$  defined by

$$(\Sigma\{b_i.f_i : i \in I\})(m) = \Sigma\{b_i.f_i(m) : i \in I\}.$$

$$\Sigma\{b_i.f_i : i \in I\} \in M[A[B]] :$$

Suppose that  $m \neq n$ . Then

$$\begin{aligned} & (\Sigma\{b_i.f_i : i \in I\})(m) \wedge (\Sigma\{b_i.f_i : i \in I\})(n) \\ &= \Sigma\{b_i.f_i(m) : i \in I\} \wedge \Sigma\{b_i.f_i(n) : i \in I\} \\ &= \Sigma\{b_i.(f_i(m) \wedge f_i(n)) : i \in I\} \\ &= \Sigma\{b_i.0 : i \in I\} = 0. \end{aligned}$$

Also,

$$\begin{aligned} & \bigvee \{(\Sigma\{b_i.f_i : i \in I\})(m) : m \in M\} \\ &= \bigvee \{\Sigma\{b_i.f_i(m) : i \in I\} : m \in M\} \\ &= \Sigma\{b_i. \bigvee \{f_i(m) : m \in M\} : i \in I\} \text{ by Proposition 2.1.3} \\ &= \Sigma\{b_i.1 : i \in I\} = 1. \end{aligned}$$

For  $H \in M[A[B]]$  and  $m \in M$ ,  $H(m)$  is a function from  $A$  into  $B$ . Define a function  $\tilde{H} : A \rightarrow B$  by

$$\tilde{H}(a) = \bigvee \{H(m)(a) : m \in M\}.$$

Suppose that  $a \neq \tilde{a}$  and  $\tilde{H}(a) \wedge \tilde{H}(\tilde{a}) > 0$ . By definition of  $\tilde{H}$ , there exist  $m, n \in M$  such that  $H(m)(a) \wedge H(n)(\tilde{a}) > 0$ . Since  $H(m)(a) \wedge H(m)(\tilde{a}) = 0$  ( $H(m) \in A[B]$ ), it follows that  $m \neq n$ . Now  $H(m) \wedge H(n) = 0$  since  $H \in M[A[B]]$ , and by Remark 0.4.5(a) and the definition of the meet operation in  $A[B]$ ,

$$0 = H(m) \wedge H(n) = \Sigma\{\Sigma\{(H(m)(x) \wedge H(n)(y)).(x \wedge y) : y \in A\} : x \in A\}.$$

Hence  $H(m)(a) \wedge H(n)(\tilde{a}) > 0$  implies that  $a \wedge \tilde{a} = 0$ . Since  $A$  satisfies the  $\kappa$ -chain condition w.r.t.  $B$ , there exists a set  $\{b_i : i \in I\}$  in  $B$  with  $\vee\{b_i : i \in I\} = 1$  and  $|\{a : \tilde{H}(a) \wedge b_i > 0\}| < \kappa$ . For each  $i \in I$  and  $a \in A$ , define a partition of  $b_i$  by

$$\{b_i \wedge H(m)(a) : m \in M\} \cup \{b_i \wedge (\tilde{H}(a))^c\}.$$

By the  $\kappa$ -chain condition, there are fewer than  $\kappa$  partitions of this kind, and since  $B$  satisfies the  $(\kappa, \infty)$ -distributive law, these partitions have a common refinement  $\{b_{ij} : j \in J\}$ . All elements of the form  $b_i \wedge H(m)(a)$  are elements of partitions of which  $\{b_{ij} : j \in J\}$  is a refinement and hence

$$H(m)(a) \wedge b_{ij} > 0 \text{ iff } H(m)(a) \geq b_{ij}.$$

Furthermore, since  $\vee\{H(m)(a) : a \in A\} = 1$ ,

$$\{a \in A : H(m)(a) \geq b_{ij}\} \neq \emptyset.$$

But since  $H(m)(a) \wedge H(m)(\tilde{a}) = 0$  if  $a \neq \tilde{a}$ , it follows that for each  $i, j$  and  $m$  there exists a unique  $a \in A$  such that  $H(m)(a) \geq b_{ij}$ . Suppose that  $f_{ij}(m)$  is the unique  $a \in A$  in the above line. Then

$$\begin{aligned} & \vee \{\vee\{\wedge\{H(m)(f_{ij}(m)) : m \in M\} : j \in J\} : i \in I\} \\ &= \vee \{\vee\{b_{ij} : j \in J\} : i \in I\} = 1. \end{aligned}$$

Hence

$$\vee \{\wedge\{H(m)(f(m)) : m \in M\} : f \in A^M\} = 1.$$

If there exist  $m, n \in M$  such that  $m \neq n$  but  $f(m) \wedge f(n) > 0$ , then by Remark 0.4.5(a) and the definition of the meet operation in  $A[B]$ ,

$$H(m)(f(m)) \wedge H(n)(f(n)) = 0,$$

and hence

$$\wedge\{H(m)(f(m)) : m \in M\} = 0.$$

Hence if  $f \notin M[A]$ , then

$$\wedge\{H(m)(f(m)) : m \in M\} = 0,$$

which implies that

$$\vee\{\wedge\{H(m)(f(m)) : m \in M\} : f \in M[A]\} = 1.$$

For each  $f \in M[A]$ , define  $b_f$  by

$$b_f = \wedge\{H(m)(f(m)) : m \in M\}.$$

$$H = \Sigma\{b_f.f : f \in M[A]\}:$$

If  $f \neq g$ , then there exists  $\tilde{m} \in M$  such that  $f(\tilde{m}) \neq g(\tilde{m})$ . Hence

$$\begin{aligned} b_f \wedge b_g &= \wedge\{H(m)(f(m)) : m \in M\} \wedge \wedge\{H(m)(g(m)) : m \in M\} \\ &\leq H(\tilde{m})(f(\tilde{m})) \wedge H(\tilde{m})(g(\tilde{m})) = 0. \end{aligned}$$

Also, since  $\vee\{b_f : f \in M[A]\} = 1$ , for each  $m \in M$  and  $a \in A$ ,

$$\begin{aligned} H(m)(a) &= H(m)(a) \wedge \vee\{b_f : f \in M[A]\} \\ &= \vee\{H(m)(a) \wedge b_f : f \in M[A]\} \\ &= \vee\{H(m)(a) \wedge \wedge\{H(n)(f(n)) : n \in M\} : f \in M[A]\}. \end{aligned}$$

If  $f(m) \neq a$ , then  $H(m)(a) \wedge H(m)(f(m)) = 0$ . Hence

$$\begin{aligned} H(m)(a) &= \vee\{\wedge\{H(n)(f(n)) : n \in M\} : f \in M[A] \text{ and } f(m) = a\} \\ &= \vee\{b_f : f \in M[A] \text{ and } f(m) = a\}, \end{aligned}$$

and the result follows by Remark 0.4.5(b).

Now for each  $f \in M[A]$ ,

$$f/F = \{g \in M[A] : \|f = g\| \in F\}.$$

Each  $H \in M[A[B]]$  is of the form  $\Sigma\{b_i.f_i : i \in I\}$  where  $\{b_i : i \in I\}$  is a pairwise disjoint set in  $B$  with  $\vee\{b_i : i \in I\} = 1$ , and  $\{f_i : i \in I\}$  is an arbitrary set in  $M[A]$ . Define a function  $K : M[A[B]] \rightarrow (M[A]/F)[B]$  by

$$K(\Sigma\{b_i.f_i : i \in I\}) = \Sigma\{b_i.f_i/F : i \in I\}.$$

Clearly,  $K$  is 1 – 1 and onto  $(M[A]/F)[B]$ , and it remains to be shown that for any formula  $\varphi$ ,

$$\|\varphi(\Sigma\{b_i.f_i : i \in I\})\| \in F \otimes G \text{ iff } \|\varphi(\Sigma\{b_i.f_i/F : i \in I\})\| \in G,$$

which would imply that  $K$  is an isomorphism.

By definition of the Boolean value,

$$\begin{aligned} \|\varphi(\Sigma\{b_i.f_i : i \in I\})\| &= \vee\{\Sigma\{b_i.f_i(m) : i \in I\} : M \models \varphi(m)\} \\ &= \Sigma\{b_i. \vee\{f_i(m) : M \models \varphi(m)\} : i \in I\} \\ &\text{by Proposition 2.1.3.} \end{aligned}$$

Hence

$$\begin{aligned} \|\varphi(\Sigma\{b_i.f_i : i \in I\})\| \in F \otimes G &\text{ iff } \Sigma\{b_i. \vee\{f_i(m) : M \models \varphi(m)\} : i \in I\} \in F \otimes G \\ &\text{ iff } \vee\{b_i : \vee\{f_i(m) : M \models \varphi(m)\} \in F\} \in G \text{ by} \\ &\text{ definition of } F \otimes G \\ &\text{ iff } \vee\{b_i : M[A]/F \models \varphi(f_i/F)\} \in G \\ &\text{ iff } \|\varphi(\Sigma\{b_i.f_i/F : i \in I\})\| \in G. \end{aligned}$$

□

Mansfield's [1971] approach to the problem of finite iteration is concluded with the following results which show that if a complete Boolean algebra  $B$  satisfies the  $(\aleph_1, \infty)$ -distributive law, then  $B$  and  $2^\omega$  satisfy the conditions for finite iteration. Mansfield [1971] requires this fact to show that elementarily equivalent models have isomorphic Boolean ultrapowers, as will be seen in Chapter 3.

**Theorem 2.1.7 [R. Mansfield 1971] :**

*Suppose that  $B$  is a complete Boolean algebra and  $B$  satisfies the  $(\aleph_1, \infty)$ -distributive law. Then  $2^\omega$  satisfies the  $\aleph_1$ -chain condition w.r.t.  $B$ .*

**Proof :**

Suppose that a function  $\tilde{H} : 2^\omega \rightarrow B$  such that

$$\tilde{H}(X_1) \wedge \tilde{H}(X_2) > 0 \text{ implies that } X_1 = X_2 \text{ or } X_1 \cap X_2 = \emptyset.$$

Suppose that  $H(n) = \vee \{\tilde{H}(X) : n \in X\}$ . Then

$$\wedge \{\vee \{(H(n))^c \vee \tilde{H}(X) : n \in X\} : n \in \omega\} = 1,$$

and since  $B$  satisfies the  $(\aleph_1, \infty)$ -distributive law,

$$\vee \{\wedge \{(H(n))^c \vee \tilde{H}(f(n)) : n \in \omega\} : n \in f(n)\} = 1.$$

Now suppose that  $b_f = \wedge \{(H(n))^c \vee \tilde{H}(f(n)) : n \in \omega\}$ . If  $b_f \wedge \tilde{H}(X) > 0$  and  $X \neq \emptyset$ , then there exists  $n \in X$  such that  $n \in f(n)$  and

$$\begin{aligned} & \tilde{H}(X) \wedge ((H(n))^c \vee \tilde{H}(f(n))) > 0 \\ \text{i.e. } & (\tilde{H}(X) \wedge (H(n))^c) \vee (\tilde{H}(X) \wedge \tilde{H}(f(n))) > 0. \end{aligned}$$

Since  $n \in X$ ,  $\tilde{H}(X) \wedge (H(n))^c = 0$ , and hence  $\tilde{H}(X) \wedge \tilde{H}(f(n)) > 0$ , which implies that  $X = f(n)$ .  $\square$

**Theorem 2.1.8 [R. Mansfield 1971] :**

*Suppose that  $B$  is a complete Boolean algebra and  $B$  satisfies the  $(\aleph_1, \infty)$ -distributive law. Then*

$$2^\omega[B] \cong B^\omega.$$

**Proof :**

Define a function  $H : 2^\omega[B] \rightarrow B^\omega$  by

$$H(f) = \langle (\vee \{f(X) : n \in X\}) : n \in \omega \rangle.$$

Clearly,  $H$  is an embedding from  $2^\omega[B]$  into  $B^\omega$ , even if  $B$  does not satisfy any distributive laws. Now suppose that  $\langle b_n : n \in \omega \rangle \in B^\omega$ . Note that  $\vee \{b_n \vee (b_n)^c : n \in \omega\} = 1$ , and if  $(b_n)^c = b_n^0$  and  $b_n = b_n^1$ , then

$$\vee \{\wedge \{b_n^{f(n)} : n \in \omega\} : f \in 2^\omega\} = 1,$$

since  $B$  satisfies the  $(\aleph_1, \infty)$ -distributive law. The pre-image of  $\langle b_n : n \in \omega \rangle$  is  $\Sigma \{\wedge \{b_n^{f(n)} : n : f(n) = 1\} : n \in \omega\} : f \in 2^\omega\}$ .  $\square$

**Corollary 2.1.9 [R. Mansfield 1971] :**

*Suppose that  $B$  is a complete Boolean algebra and  $B$  satisfies the  $(\aleph_1, \infty)$ -distributive law. Then  $2^\omega[B]$  is complete.*

## 2.2 The approach of P. Ouwehand and H. Rose [1998] to the problem of finitely iterated Boolean ultrapowers

The approach of Ouwehand and Rose [1998] is introduced by a discussion on some general results concerning direct limits and elementary embeddings on Boolean ultrapowers.

Suppose that  $\mathcal{U}$  is an updirected set and  $\{B_P : P \in \mathcal{U}\}$  is a family of complete Boolean algebras such that if  $P \leq Q$  (i.e.  $Q$  is a refinement of  $P$  according to the partial ordering imposed on  $\mathbb{P}_B$ ), then  $B_P \subseteq B_Q$ . Suppose that  $F_P$  is an ultrafilter on  $B_P$  for each  $P \in \mathcal{U}$ , and that the family  $\{F_P : P \in \mathcal{U}\}$  is updirected. For  $P \leq Q$ , define a function  $j_{PQ} : M[B_P]_\kappa / F_P \rightarrow M[B_Q]_\kappa / F_Q$  by

$$j_{PQ}(f/F_P) = f/F_Q.$$

**Lemma 2.2.1 :**

*$j_{PQ}$  is an elementary embedding.*

**Proof :**

$j_{PQ}$  is a well-defined embedding:

Suppose that  $f, g \in M[B_P]_\kappa$ . Then

$$\begin{aligned} f/F_P = g/F_P & \text{ iff } \|f = g\| \in F_P \\ & \text{ iff } \vee \{s \in S : f(s) = g(s)\} \in F_P \\ & \text{ where } S \text{ is a common domain of } f \text{ and } g \\ & \text{ iff } \vee \{s \in S : f(s) = g(s)\} \in F_Q \\ & \text{ iff } \|f = g\| \in F_Q \\ & \text{ iff } f/F_Q = g/F_Q. \end{aligned}$$

$j_{PQ}$  is elementary :

Suppose that  $\varphi$  is a first-order formula and  $f_1, \dots, f_n \in M[B_P]_\kappa$ .

Suppose that

$$M[B_P]_\kappa / F_Q \models \exists x \varphi(x, f_1/F_Q, \dots, f_n/F_Q).$$

Hence there exists  $g \in M[B_P]_\kappa$  such that

$$M[B_P]_\kappa / F_Q \models \varphi(g/F_Q, f_1/F_Q, \dots, f_n/F_Q).$$

Then

$$\forall \{s \in S : M[B_P]_\kappa \models \varphi(g(s), f_1(s), \dots, f_n(s))\} \in F_Q \text{ where}$$

$S$  is a common domain of  $g, f_1, \dots, f_n$ , and hence

$$\forall \{s \in S : M[B_P]_\kappa \models \varphi(g(s), f_1(s), \dots, f_n(s))\} \in F_P,$$

which implies that

$$M[B_P]_\kappa / F_P \models \varphi(g/F_P, f_1/F_P, \dots, f_n/F_P).$$

The result follows by Lemma 0.1.8 (Tarski-Vaught Criterion).  $\square$

Now define a model  $N$  to be the direct limit of the models  $\{M[B_P]_\kappa / F_P : P \in \mathcal{U}\}$ . More precisely, consider  $\cup\{M[B_P]_\kappa / F_P : P \in \mathcal{U}\}$  and define an equivalence relation  $\sim$  on  $\cup\{M[B_P]_\kappa / F_P : P \in \mathcal{U}\}$  as follows: For  $f \in M[B_P]_\kappa$ ,  $g \in M[B_Q]_\kappa$ ,

$$f/F_P \sim g/F_Q \text{ iff there exists } T \geq P, Q \text{ such that } j_{PT}(f/F_P) = j_{QT}(g/F_Q).$$

$N$  is then the set of all equivalence classes of  $\cup\{M[B_P]_\kappa / F_P : P \in \mathcal{U}\}$  w.r.t. the embeddings  $j_{PQ}$ .

The functions  $\eta_P : M[B_P]_\kappa / F_P \rightarrow N$  defined by

$$\eta_P(f/F_P) = (f/F_P) / \sim$$

are elementary embeddings by a similar application of the Tarski-Vaught Criterion as in Lemma 2.2.1. Suppose that  $B$  is the direct limit of the family  $\{B_P : P \in \mathcal{U}\}$ , and that  $F = \cup\{F_P : P \in \mathcal{U}\}$ . By Lemma 0.2.4.7(a),  $F$  is an ultrafilter on  $B$ , and hence there exist elementary embeddings  $m_P : M[B_P]_\kappa / F_P \rightarrow M[B]_\kappa / F$  defined by

$$m_P(f/F_P) = f/F.$$

Thus there exists a commutative diagram of elementary embeddings, and by the property of the direct limit, the function  $e : N \rightarrow M[B]_\kappa / F$  exists, where  $e$  is defined by

$$e((f/F_P) / \sim) = f/F.$$



Recall (Definition 0.2.4.5) that a  $\kappa$ -complete Boolean algebra  $B$  has the  $\kappa$ -partition cofinality property w.r.t. the family  $\{B_P : P \in \mathcal{U}\}$  of complete Boolean algebras if for each  $Q \in \mathbb{P}_B^\kappa$  there exist  $P \in \mathcal{U}$  and a partition  $T$  of  $B_P$  such that  $Q \leq T$ .

**Theorem 2.2.2 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $B$  has the  $\kappa$ -partition cofinality property w.r.t. the family  $\{B_P : P \in \mathcal{U}\}$  of complete Boolean algebras. Then*

$$M[B]_\kappa / F \cong N$$

*i.e. the Boolean ultrapower of the direct limit is isomorphic to the direct limit of the Boolean ultrapowers.*

**Proof :**

Since  $B$  has the  $\kappa$ -partition cofinality property w.r.t. the family  $\{B_P : P \in \mathcal{U}\}$ ,  $B$  is  $\kappa$ -complete by Lemma 0.2.4.6. Hence  $M[B]_\kappa$  is defined, and it remains to be shown that the elementary embedding  $e : N \rightarrow M[B]_\kappa / F$  is surjective. Suppose that  $f : Q \rightarrow M$  for  $Q \in \mathbb{P}_B^\kappa$ . By the  $\kappa$ -partition cofinality property of  $B$  w.r.t.  $\{B_P : P \in \mathcal{U}\}$ , there exist  $P \in \mathcal{U}$  and a partition  $T \in \mathbb{P}_{B_P}^\kappa$  such that  $T$  refines  $Q$ . Then  $f : T \rightarrow M$  since  $T$  refines  $Q$ , and hence  $f \in M[B_P]_\kappa$ . Clearly,  $f/F$  has a pre-image  $(f/F) / \sim$  under  $e$ .  $\square$

The Finite Iteration Theorem for  $\kappa$ -bounded Boolean ultrapowers of Ouwehand and Rose [1998] will now be proved.

Suppose that  $A$  and  $B$  are  $\kappa$ -complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $A$  and  $B$  respectively. Recall that, by Lemma 0.2.4.2(b),  $A$  is the direct limit of powerset Boolean algebras, and by Lemma 0.2.4.7(b),  $\{F_P = F \cap \mathcal{P}(P) : P \in \mathbb{P}_A^\kappa\}$  is an updirected family of ultrafilters on  $\mathcal{P}(P)$ . Similar statements apply to  $B$  and  $G$ .

**Corollary 2.2.3 [P. Ouwehand and H. Rose 1998] :**

*Any  $\kappa$ -bounded Boolean ultrapower  $M[A]_\kappa / F$  is the direct limit of the ordinary ultrapowers  $\{M^P / F_P : P \in \mathbb{P}_A^\kappa\}$ .*

**Proof :**

By Theorem 2.2.2, it suffices to show that  $A$  has the  $\kappa$ -partition cofinality property w.r.t. the family  $\{\mathcal{P}(P) : P \in \mathbb{P}_A^\kappa\}$ . Suppose that  $T \in \mathbb{P}_A^\kappa$ . Then  $T$  is a partition of  $\mathcal{P}(T)$ , and hence  $M[A]_\kappa / F$  is isomorphic to the direct limit of

$\{M[\mathcal{P}(P)]_\kappa/F_P : P \in \mathbb{P}_A^\kappa\}$ . However, since  $|P| < \kappa$  for each  $P \in \mathbb{P}_A^\kappa$ , it follows that

$$M[\mathcal{P}(P)]_\kappa/F_P = M[\mathcal{P}(P)]/F_P \cong M^P/F_P.$$

□

Now by Lemma 2.1.4 and the remark following it, it follows that a  $\kappa$ -complete Boolean algebra  $B$  satisfies the  $(\kappa, \kappa)$ -distributive law iff  $X \subseteq \mathbb{P}_B^\kappa$  such that  $|X| < \kappa$ , implies that the partitions in  $X$  have a common refinement. However, this refinement need not be in  $\mathbb{P}_B^\kappa$ . Consequently, Jipsen and Rose [1999] introduce the following notion which is stronger than the notion of  $(\kappa, \kappa)$ -distributivity, and is essential to the approach of Ouwehand and Rose to the problem of finitely iterated Boolean ultrapowers.

**Definition 2.2.4 :**

A  $\kappa$ -complete Boolean algebra  $B$  is  *$\kappa$ -partition complete* if for each set  $X \subseteq \mathbb{P}_B^\kappa$  such that  $|X| < \kappa$ , the partitions in  $X$  have a common refinement in  $\mathbb{P}_B^\kappa$ .

**Examples 2.2.5 :**

- (a) If  $|Y| < \kappa$ , then the powerset Boolean algebra  $\mathcal{P}(Y)$  is  $\kappa$ -partition complete since it is completely distributive and  $\mathbb{P}_{\mathcal{P}(Y)}^\kappa = \mathbb{P}_{\mathcal{P}(Y)}$ .
- (b) It is well-known that any  $\kappa$ -complete subalgebra of a powerset Boolean algebra satisfies the  $(\kappa, \kappa)$ -distributive law. Jipsen and Rose [1999] show that if  $\kappa$  is strongly inaccessible, then the notions of  $(\kappa, \kappa)$ -distributivity and  $\kappa$ -partition completeness coincide. Hence if  $\kappa$  is strongly inaccessible, then any  $\kappa$ -complete subalgebra of a powerset Boolean algebra is  $\kappa$ -partition complete.
- (c) A complete atomless Boolean algebra  $B$  is called a  *$\kappa$ -Suslin algebra* if it is  $(\kappa, \kappa)$ -distributive and every pairwise disjoint subset has fewer than  $\kappa$  elements. Then  $\mathbb{P}_B^\kappa = \mathbb{P}_B$ , and hence  $B$  is  $\kappa$ -partition complete.

The following observation shows that if, in addition,  $A$  is  $\kappa$ -partition complete, then there exist a  $\kappa$ -complete Boolean algebra  $C$  and an ultrafilter  $E$  on  $C$  such that

$$(M[A]_\kappa/F)[B]_\kappa/G \cong M[C]_\kappa/E.$$

By Corollary 2.2.3,

$$\begin{aligned} (M[A]_\kappa/F)[B]_\kappa/G &\cong \bigcup \{(M[A]_\kappa/F)^Q/G_Q : Q \in \mathbb{P}_B^\kappa\} \\ &\cong \bigcup \{(\bigcup \{M^P/F_P : P \in \mathbb{P}_A^\kappa\})^Q/G_Q : Q \in \mathbb{P}_B^\kappa\} \\ &\cong \bigcup \{M^{P \times Q}/F_P \otimes G_Q : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\} \\ &\text{since } A \text{ is } \kappa\text{-partition complete.} \end{aligned}$$

This suggests that  $C$  be defined to be the direct limit of the powerset Boolean algebras  $\{\mathcal{P}(P \times Q) : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$ , and  $E$  to be  $\cup\{F_P \otimes G_Q : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$ . However, in order for  $C$  and  $E$  to be defined as such,  $C$  has to have the  $\kappa$ -partition cofinality property w.r.t. the family  $\{\mathcal{P}(P \times Q) : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$ , and the family of ultrafilters  $\{F_P \otimes G_Q : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$  has to be updirected. This will be shown in the following two lemmas.

Suppose that  $P$  and  $Q$  are refinements of  $\tilde{P}$  and  $\tilde{Q}$  respectively. Define complete embeddings  $i : \mathcal{P}(\tilde{P} \times \tilde{Q}) \rightarrow \mathcal{P}(P \times Q)$  by

$$i : U \subseteq \tilde{P} \times \tilde{Q} \rightarrow \{(p, q) \in P \times Q : \exists(\tilde{p}, \tilde{q}) \in U[(p, q) \leq (\tilde{p}, \tilde{q})]\}.$$

**Lemma 2.2.6 [P. Ouwehand and H. Rose 1998] :**

*The family of ultrafilters  $\{F_P \otimes G_Q : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$  is updirected.*

**Proof :**

Suppose that  $P$  and  $Q$  are refinements of  $\tilde{P}$  and  $\tilde{Q}$  respectively, and  $U \in F_{\tilde{P}} \otimes G_{\tilde{Q}}$ . Then

$$\{\tilde{q} \in \tilde{Q} : \{\tilde{p} \in \tilde{P} : (\tilde{p}, \tilde{q}) \in U\} \in F_{\tilde{P}}\} \in G_{\tilde{Q}}, \text{ and thus}$$

$$\{q \in Q : \exists \tilde{q} \in \tilde{Q}[q \leq \tilde{q} \text{ and } \{\tilde{p} \in \tilde{P} : (\tilde{p}, \tilde{q}) \in U\} \in F_{\tilde{P}}]\} \in G_Q,$$

which implies that

$$\{q \in Q : \exists \tilde{q} \in \tilde{Q}[q \leq \tilde{q} \text{ and } \{p \in P : \exists \tilde{p} \in \tilde{P}[p \leq \tilde{p} \text{ and } (\tilde{p}, \tilde{q}) \in U]\} \in F_{\tilde{P}}]\} \in G_Q.$$

Hence

$$\{q \in Q : \{p \in P : \exists(\tilde{p}, \tilde{q}) \in U[(p, q) \leq (\tilde{p}, \tilde{q})]\} \in F_{\tilde{P}}\} \in G_Q,$$

i.e.  $i(U) \in F_P \otimes G_Q$ .

**Lemma 2.2.7 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $C$  is the direct limit of the family  $\{\mathcal{P}(P \times Q) : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$  of powerset Boolean algebras. Then  $C$  has the  $\kappa$ -partition cofinality property w.r.t.  $\{\mathcal{P}(P \times Q) : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$ .*

**Proof :**

Recall that  $C$  is a set of equivalence classes  $\{U / \sim : U \subseteq P \times Q \text{ and } (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$  with the property that if  $U_1 \subseteq P_1 \times Q_1$  and  $U_2 \subseteq P_2 \times Q_2$ , then  $U_1 \sim U_2$  iff there exists  $(P_3, Q_3) \geq (P_1, Q_1), (P_2, Q_2)$  such that for all  $(p, q) \in P_3 \times Q_3$ ,

$$\exists(\tilde{p}, \tilde{q}) \in U_1[(p, q) \leq (\tilde{p}, \tilde{q})] \text{ iff } \exists(\tilde{p}, \tilde{q}) \in U_2[(p, q) \leq (\tilde{p}, \tilde{q})].$$

(This follows from the definition of the complete embeddings  $i : \mathcal{P}(\tilde{P} \times \tilde{Q}) \rightarrow \mathcal{P}(P \times Q)$ .)

Suppose that  $\{U_i / \sim : i \in I\}$  is a partition of  $C$ . W.l.o.g. assume that  $U_i \subseteq P_i \times Q_i \subseteq A \times B$  for each  $i \in I$ . Define

$$U_{i1} = \{p \in A : \exists q \in B[(p, q) \in U_i]\},$$

$$U_{i2} = \{q \in B : \exists p \in A[(p, q) \in U_i]\}.$$

Now suppose that  $(p_i, q_i) \in U_i \subseteq P_i \times Q_i$  and  $(p_j, q_j) \in U_j \subseteq P_j \times Q_j$ , and suppose that  $P$  and  $Q$  are the coarsest common refinements of  $P_i$  and  $P_j$  and  $Q_i$  and  $Q_j$  respectively. Since  $U_i / \sim \wedge U_j / \sim = 0$ , if  $(p, q) \in P \times Q$  such that  $(p, q) \leq (p_i, q_i)$ , then  $(p, q) \not\leq (p_j, q_j)$ . Similarly, if  $(p, q) \in P \times Q$  such that  $(p, q) \leq (p_j, q_j)$ , then  $(p, q) \not\leq (p_i, q_i)$ . But

$$p_i = \vee\{p \in P : p \leq p_i\}, \quad \text{and}$$

$$p_j = \vee\{p \in P : p \leq p_j\}.$$

Since no  $p \in P$  is such that  $p \leq p_i$  and  $p \leq p_j$ , it follows that

$$p_i \wedge p_j = \vee\{p \wedge \tilde{p} : p, \tilde{p} \in P \text{ and } p \leq p_i, \tilde{p} \leq p_j\} = 0.$$

Similarly,  $q_i \wedge q_j = 0$ . Hence  $\cup\{U_{i1} : i \in I\}$  and  $\cup\{U_{i2} : i \in I\}$  are pairwise disjoint subsets of  $A$  and  $B$  respectively, which implies that there exist partitions  $P \in \mathbb{P}_A^\kappa$  and  $Q \in \mathbb{P}_B^\kappa$  such that  $\cup\{U_{i1} : i \in I\} \subseteq P$  and  $\cup\{U_{i2} : i \in I\} \subseteq Q$ . Then  $U_i \subseteq P \times Q$ . But  $\mathcal{P}(P \times Q)$  has a finest partition  $\{(p, q) : (p, q) \in P \times Q\}$ , which is clearly finer than  $\{U_i / \sim : i \in I\}$ . Hence  $C$  has the  $\kappa$ -partition cofinality property w.r.t.  $\{\mathcal{P}(P \times Q) : (P, Q) \in \mathbb{P}_A^\kappa \times \mathbb{P}_B^\kappa\}$ .

The following theorem has thus been proved.

**Theorem 2.2.8 [P. Ouwehand and H. Rose 1998] :**

*Suppose that  $A$  and  $B$  are  $\kappa$ -complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $A$  and  $B$  respectively. If  $A$  is  $\kappa$ -partition complete, then the iterated  $\kappa$ -bounded Boolean ultrapower  $(M[A]_\kappa / F)[B]_\kappa / G$  is isomorphic to a single  $\kappa$ -bounded Boolean ultrapower.*

**Remarks 2.2.9 :**

- (a) If  $A$  and  $B$  are  $\kappa$ -partition complete, then by Lemma 0.5 in Jipsen and Rose [1999],  $A$  and  $B$  are  $\kappa$ -complete, and hence Theorem 2.2.8 holds.
- (b) If  $A = C_\kappa$  is the completion of a  $(\kappa, \kappa)$ -tree and  $\kappa$  is strongly inaccessible, then the lattice  $\mathbb{P}_A$  is  $\kappa$ -complete and  $\mathbb{P}_A^\kappa = \mathbb{P}_A$ . By Proposition 0.6 in Jipsen and Rose [1999],  $A$  is  $\kappa$ -partition complete and Theorem 2.2.8 holds.

**Problems 2.2.10 :**

- (a) Does the Finite Iteration Theorem hold for arbitrary Boolean ultrapowers? (Does a counter-example exist?) Find necessary and sufficient conditions for finite iteration or show that it is impossible to find such conditions.
  - (b) Define the notion of an infinitely iterated Boolean ultrapower. Develop the theory of infinitely iterated Boolean ultrapowers.
-

## Chapter 3

### Elementarily equivalent models have isomorphic Boolean ultrapowers

The Keisler - Shelah Theorem (see Hodges [1993]) states that any two models of the same type are elementarily equivalent iff they have isomorphic ultrapowers. The sufficiency part of the theorem is an immediate consequence of Łos's Theorem. Mansfield [1971] obtains an analogous result for Boolean ultrapowers. Mansfield shows that if any two models are elementarily equivalent and if in addition, they are  $\aleph_1$ -saturated, then they have isomorphic Boolean ultrapowers. He achieves this by constructing an inner product into an appropriately defined Boolean algebra and then deducing that the models have isomorphic Boolean powers, and hence they have isomorphic Boolean ultrapowers. Finally, by applying the Finite Iteration Theorem for Boolean ultrapowers and the often-used result that a Boolean ultrapower is  $\aleph_1$ -saturated if the ultrafilter is descendently countably incomplete, he concludes that any two elementarily equivalent models have isomorphic Boolean ultrapowers. The converse is immediate, since by the Boolean version of Łos's Theorem, any model is elementarily embeddable into any of its Boolean ultrapowers.

In this chapter a detailed discussion on Mansfield's above-mentioned result will be presented. Mansfield uses Foster's [1953] definition of a Boolean power for his proof.

A model  $M$  is  $\kappa$ -saturated if whenever  $\Gamma$  is a set of formulas with one free variable which is finitely satisfiable and  $|\Gamma| < \kappa$ , then  $\Gamma$  is also satisfiable.

**Definition 3.1 [K. Kunen and K. Prikry 1971] :**

An ultrafilter  $F$  is *descendently countably incomplete* if there exists a descending sequence  $\langle b_n \in F : n \in \omega \rangle$  such that  $\bigwedge \{b_n : n \in \omega\} = 0$ .

The following theorem is Mansfield's Boolean version of Theorem 6.1.1 in Chang and Keisler [1973].

**Theorem 3.2 [R. Mansfield 1971] :**

*If  $F$  is descendently countably incomplete, then the Boolean ultrapower  $M[B]/F$  is  $\aleph_1$ -saturated.*

**Proof :**

Suppose that  $\{\varphi_n(x) : n \in \omega\}$  is a set of formulas of one free variable. By taking conjunctions if necessary, w.l.o.g.  $\varphi_n$  is a logical consequence of  $\varphi_{n+1}$

i.e.  $\|\varphi_n\| \geq \|\varphi_{n+1}\|$ . Now suppose that  $\langle b_n : n \in \omega \rangle$  is a descending sequence in  $F$ , and suppose that  $\tilde{b}_n = b_n \wedge \|\exists x \varphi_n(x)\|$ . Then for each  $n \in \omega$ ,  $\tilde{b}_n \in F$ , and  $\bigwedge \{\tilde{b}_n : n \in \omega\} \leq \bigwedge \{b_n : n \in \omega\} = 0$  since  $F$  is descendingly countably incomplete. Define  $\hat{b}_n$  by

$$\hat{b}_n = \tilde{b}_n \wedge (\tilde{b}_{n+1})^c.$$

By Theorem 0.4.6, there exists  $f_n \in M[B]$  for each  $n \in \omega$  such that

$$\|\exists x \varphi_n(x)\| = \|\varphi_n(f_n)\|.$$

Now  $\{\hat{b}_n : n \in \omega\}$  is a pairwise disjoint set since  $\langle \tilde{b}_n : n \in \omega \rangle$  is a disjoint sequence. Hence by Theorem 0.4.4, there exists  $f \in M[B]$  such that  $\|f = f_n\| \geq \hat{b}_n$  for each  $n \in \omega$ . Then for  $k \geq n$ ,

$$\begin{aligned} \|\varphi_n(f)\| &\geq \|\varphi_n(f_k)\| \wedge \|f = f_k\| \\ &\geq \|\varphi_k(f_k)\| \wedge \hat{b}_k \\ &\geq \hat{b}_k. \end{aligned}$$

Hence

$$\|\varphi_n(f)\| \geq \bigvee \{\hat{b}_k : k \geq n\} = \tilde{b}_n \in F,$$

and the result follows.  $\square$

### Definition 3.3 :

Suppose that  $M$  and  $N$  are two models of the same type. An *inner product* on  $M \times N$  is a function  $(.,.) : M \times N \rightarrow B$  where  $B$  is a complete Boolean algebra, such that

- (a)  $\bigwedge \{(m_i, n_i) : 1 \leq i \leq k\} > 0$  iff  $(M, m_1, \dots, m_k) \equiv (N, n_1, \dots, n_k)$ ,
- (b)  $\bigvee \{(m, n) : m \in M\} = \bigvee \{(m, n) : n \in N\} = 1$ .

For the discussion that follows, suppose that  $M$  and  $N$  are infinite elementarily equivalent  $\aleph_1$ -saturated models. In order to construct an inner product on  $M \times N$ , the complete Boolean algebra  $B$  into which  $M \times N$  is mapped needs to be defined.

Suppose that  $f$  is a partial function from  $M$  into  $N$  such that  $(M, \text{dom } f) \equiv (N, \text{range } f)$  and denote this by  $f : M \xrightarrow{P} N$ . Suppose that

$$X = \{f \mid f : M \xrightarrow{P} N \text{ and } |f| = \aleph_1\}.$$

If  $g$  is a partial function from  $M$  into  $N$  and  $g$  is countable, then denote the set  $\{f \in X : g \subseteq f\}$  by  $[g]$ . Now the set

$$\{[g] \mid g : M \xrightarrow{P} N \text{ and } |g| \leq \aleph_0\}$$

is a base for a topology on  $X$ . Define  $B$  by

$$B = RO(X), \text{ the regular open algebra on } X.$$

**Lemma 3.4 [R. Mansfield 1971] :**

*Suppose that the language of  $M$  and  $N$  is countable. Then  $[g] \neq \emptyset$  iff  $g : M \xrightarrow{P} N$ .*

**Proof :**

Suppose that  $g : M \xrightarrow{P} N$  and  $|g| \leq \aleph_0$ . Now the set  $\{f \mid f : M \xrightarrow{P} N, g \subseteq f \text{ and } |f| \leq \aleph_0\}$  is partially ordered by inclusion, and hence has a maximal chain. Suppose that this chain is countable. If  $\tilde{g}$  is the union of this chain, then  $\tilde{g}$  is countable and  $\tilde{g} : M \xrightarrow{P} N$ . Suppose that  $n \in N - \text{range } \tilde{g}$  and  $\Gamma$  is a set of formulas  $\varphi(x)$  of one free variable and constants from the range of  $\tilde{g}$  such that  $N \models \varphi(n)$ .  $\Gamma$  is countable and finitely satisfiable in  $M$ . Since  $M$  is  $\aleph_1$ -saturated, there exists  $m \in M$  such that  $M \models \varphi(m)$  for each  $\varphi \in \Gamma$ . Hence  $\tilde{g} \cup \{\langle m, n \rangle\}$  extends  $\tilde{g}$ , contradicting the assumption that the chain is countable. Hence the chain is uncountable and its union is an element of  $[g]$ . The converse is immediate.  $\square$

**Lemma 3.5 [R. Mansfield 1971] :**

*If  $\langle [g_n] : n \in \omega \rangle$  is a decreasing sequence of base sets, then  $\bigcap \{[g_n] : n \in \omega\} \neq \emptyset$ .*

**Proof :**

$[g_{n+1}] \subseteq [g_n]$  iff  $g_n \subseteq g_{n+1}$ . Hence  $g = \bigcup \{g_n : n \in \omega\}$  is countable and  $g : M \xrightarrow{P} N$ . Now since  $(M, \text{dom } g_n) \equiv (N, \text{range } g_n)$ , it follows that  $(M, \text{dom } g) \equiv (N, \text{range } g)$ , and hence  $g \in \bigcap \{[g_n] : n \in \omega\}$ .  $\square$

The following result is a standard consequence of Lemma 3.5.

**Corollary 3.6 :**

*Suppose that  $X = \{f \mid f : M \xrightarrow{P} N \text{ and } |f| = \aleph_1\}$ , and  $\{[g] \mid g : M \xrightarrow{P} N \text{ and } |g| \leq \aleph_0\}$  is a base for a topology on  $X$ . Then  $B = RO(X)$  satisfies the  $(\aleph_1, \infty)$ -distributive law.*



The inner product on  $M \times N$  is constructed as follows.

**Theorem 3.7 [R. Mansfield 1971] :**

Suppose that the function  $(.,.)$  on  $M \times N$  is defined by

$$(m, n) = \{f \in X : f(m) = n\}.$$

Then  $(.,.)$  is a  $B$ -valued inner product on  $M \times N$ .

**Proof :**

The set  $\{f \in X : f(m) = n\}$  is a clopen subset of  $X$  and is hence an element of  $B$ . Also,

$$\bigwedge \{(m_i, n_i) : 1 \leq i \leq k\} = \bigcap \{(m_i, n_i) : 1 \leq i \leq k\},$$

and  $\bigcap \{(m_i, n_i) : 1 \leq i \leq k\} \neq \emptyset$  iff there exists  $f \in X$  such that  $f(m_i) = n_i$  for each  $1 \leq i \leq k$ . Hence  $(.,.)$  satisfies condition (a) of the definition of an inner product. To show that  $(.,.)$  satisfies condition (b), suppose that  $\bigvee \{(m, n) : m \in M\} < 1$  i.e.  $\bigcup \{(m, n) : m \in M\} \neq X$ . Then there exists a base set  $[g]$  such that

$$[g] \cap \bigcup \{(m, n) : m \in M\} = \emptyset.$$

Now  $\bigcup \{(m, n) : m \in M\}$  is the set  $\{f \in X : n \in \text{range } f\}$ . Suppose that  $\Gamma$  is the set of formulas  $\varphi(x)$  of one free variable and constants from the range of  $g$  such that for each  $\varphi(x) \in \Gamma$ ,  $N \models \varphi(n)$ .  $\Gamma$  is finitely satisfiable in  $M$ , and since  $M$  is  $\aleph_1$ -saturated, there exists  $m \in M$  such that  $M \models \varphi(m)$  for each  $\varphi(x) \in \Gamma$ . Then  $\tilde{g} = g \cup \{(m, n)\}$  is countable and  $\tilde{g} : M \xrightarrow{P} N$ .  $[\tilde{g}] \subseteq [g]$ , contradicting the assumption that  $[g] \cap \bigcup \{(m, n) : m \in M\} = \emptyset$ . The proof of  $\bigvee \{(m, m) : n \in N\} = 1$  is similar.  $\square$

**Theorem 3.8 [R. Mansfield 1971] :**

If there exists a  $B$ -valued inner product on  $M \times N$ , then  $M[B] \cong N[B]$ , and hence for any ultrafilter  $F$  on  $B$ ,  $M[B]/F \cong N[B]/F$ .

**Proof :**

Define a function  $H : M[B] \rightarrow N[B]$  by

$$H(f)(n) = \bigvee \{(m, n) : m \in M\} \wedge f(m)$$

and a function  $K : N[B] \rightarrow M[B]$  by

$$K(g)(m) = \bigvee \{(m, n) : n \in N\} \wedge g(n).$$

It suffices to show that  $K \circ H$  is the identity function on  $M[B]$  and that  $H$  preserves Boolean values.

For  $\tilde{m} \in M$ ,

$$\begin{aligned} K(H(f))(\tilde{m}) &= \vee\{(\tilde{m}, n) : n \in N\} \wedge H(f)(n) \\ &= \vee\{(\tilde{m}, n) \wedge \vee\{(m, n) : m \in M\} \wedge f(m) : n \in N\} \\ &= \vee\{\vee\{(\tilde{m}, n) \wedge (m, n) \wedge f(m) : m \in M\} : n \in N\}. \end{aligned}$$

Since  $(\tilde{m}, n) \wedge (m, n) = 0$  if  $\tilde{m} \neq m$ ,

$$\begin{aligned} K(H(f))(\tilde{m}) &= \vee\{(\tilde{m}, n) \wedge f(\tilde{m}) : n \in N\} \\ &= f(\tilde{m}) \wedge \vee\{(\tilde{m}, n) : n \in N\} \\ &= f(\tilde{m}). \end{aligned}$$

Now for any formula  $\varphi$ , by Theorem 0.4.2,

$$\begin{aligned} \|\varphi(H(f))\| &= \vee\{H(f)(n) : N \models \varphi(n)\} \\ &= \vee\{\vee\{(m, n) : m \in M\} \wedge f(m) : N \models \varphi(n)\}. \end{aligned}$$

If  $(m, n) \neq 0$ , then  $M \models \varphi(m)$  iff  $N \models \varphi(n)$ , and hence

$$\begin{aligned} \|\varphi(H(f))\| &= \vee\{\vee\{(m, n) : M \models \varphi(m)\} \wedge f(m) : n \in N\} \\ &= \vee\{\vee\{(m, n) : n \in N\} \wedge f(m) : M \models \varphi(m)\} \\ &= \vee\{f(m) : M \models \varphi(m)\} = \|\varphi(f)\|. \end{aligned}$$

□

**Corollary 3.9 [R. Mansfield 1971] :**

*If two infinite models  $M$  and  $N$  are elementarily equivalent, then they have isomorphic Boolean ultrapowers.*

**Proof :**

Suppose that  $F$  is a non-principal filter on  $2^\omega$ . Then  $F$  is descendingly countably incomplete by Corollary 1.10 in Bell and Slomson [1971]. By Theorem 3.2,  $M[2^\omega]/F$  and  $N[2^\omega]/F$  are  $\aleph_1$ -saturated. Since  $M$  and  $N$  are elementarily embeddable into  $M[2^\omega]/F$  and  $N[2^\omega]/F$  respectively,  $M[2^\omega]/F \equiv N[2^\omega]/F$ . By Theorem 3.7 and Corollary 3.6, there exists an inner product on  $M \times N$  into a complete Boolean algebra  $B$  satisfying the  $(\aleph_1, \infty)$ -distributive law. By

Theorem 2.1.7 and Corollary 2.1.9,  $2^\omega$  and  $B$  satisfy the conditions for finite iteration.

Hence if  $G$  is any ultrafilter on  $B$ , then

$$\begin{aligned} M[2^\omega[B]]/F \otimes G &\cong (M[2^\omega]/F)[B]/G \text{ by Theorem 2.1.6} \\ &\cong (N[2^\omega]/F)[B]/G \text{ by Theorem 3.8} \\ &\cong (N[2^\omega[B]]/F) \otimes G \text{ by Theorem 2.1.6.} \end{aligned}$$

□

**Problem 3.10 :**

Suppose that  $M$  and  $N$  are elementarily equivalent  $\aleph_1$ -saturated models, and there exists a  $B$ -valued inner product on  $M \times N$ . By Theorem 3.8,  $M[B]/F \cong N[B]/F$ . Are  $M[B]/F$  and  $N[B]/F$  isomorphic to ordinary ultrapowers? If yes, then Theorem 3.8 provides an alternative proof of the Keisler-Shelah Theorem. If not, then  $M[B]/F$  and  $N[B]/F$  are further examples of Boolean ultrapowers which are not isomorphic to any ordinary ultrapowers.

---

## Chapter 4

### Good ultrafilters and saturated Boolean ultrapowers

A well-known result of Keisler (see Bell and Slomson [1971]) states that an ultrapower  $M^I/F$  is saturated for all models  $M$  iff the ultrafilter is good. By defining the notion of goodness for an ultrafilter on a Boolean algebra in a manner analogous to the conventional definition, Mansfield [1971] is able to mimic Keisler's argument and prove the above implication from right to left for Boolean ultrapowers. Mansfield carefully chooses a Boolean algebra in order to obtain a good ultrafilter and a saturated Boolean ultrapower. This construction incidentally provides an example of a Boolean ultrapower of inaccessible cardinality. Benda [1974] observes that Mansfield's definition is not particularly useful in proving the reverse implication of Keisler's theorem for Boolean ultrapowers. He thus proposes an alternative definition for the notion of goodness for an ultrafilter on a Boolean algebra which enables him to obtain a complete generalisation of Keisler's theorem. This definition of goodness coincides with the usual one if the Boolean algebra is atomic, and is weaker if the Boolean algebra is atomless.

In this chapter a detailed exposition on the work of Mansfield [1971] and Benda [1974] will be given. While Mansfield uses Foster's [1953] definition of a Boolean power, Benda uses a different definition of a Boolean power for his investigation. The Boolean powers resulting from the definitions of Foster and Benda are isomorphic (see Chapter 0, Section 0.2).

#### 4.1 The notion of goodness of an ultrafilter as defined by R. Mansfield [1971] and his partial generalisation of Keisler's theorem

Suppose that  $\lambda$  is an infinite cardinal and  $\mathcal{P}_\omega(\lambda)$  denotes the finite subsets of  $\lambda$ . A function  $f : \mathcal{P}_\omega(\lambda) \rightarrow B$  where  $B$  is a Boolean algebra, is *monotonically decreasing* if  $u \subseteq v$  implies that  $f(v) \leq f(u)$ .  $f$  is *multiplicative* if  $f(u \cup v) = f(u) \wedge f(v)$ .

##### Definition 4.1.1 [R. Mansfield 1971] :

An ultrafilter  $F$  on a Boolean algebra  $B$  is  $\kappa$ -*good* if it is descendingly countably incomplete, and for each cardinal  $\lambda < \kappa$  and for each monotonically decreasing function  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$ , there exists a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  such that  $g \leq f$ .

##### Theorem 4.1.2 [R. Mansfield 1971] :

*If  $F$  is  $\kappa$ -good, then  $M[B]/F$  is  $\kappa$ -saturated.*

**Proof :**

Suppose that  $\lambda < \kappa$  and  $\{\varphi_\alpha(x) : \alpha < \lambda\}$  is a set of formulas of one free variable which is finitely satisfiable in  $M[B]/F$ . Since  $F$  is descendingly countably incomplete, there exists a decreasing sequence  $\langle b_n : n \in \omega \rangle$  in  $F$  such that  $\bigwedge \{b_n : n \in \omega\} = 0$ . For  $u \in \mathcal{P}_\omega(\lambda)$ , define a function  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$  by

$$f(u) = b_{|u|} \vee \|\exists x \wedge \{\varphi_\alpha(x) : \alpha \in u\}\|.$$

Since  $f$  is monotonically decreasing and  $F$  is  $\kappa$ -good, there exists a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  such that  $g \leq f$ . Define

$$\tilde{b}_u = g(u) \wedge (\vee \{g(v) : |u| < |v|\})^c.$$

Now the set  $\{\tilde{b}_u : u \in \mathcal{P}_\omega(\lambda)\}$  is pairwise disjoint :

If  $u \not\subseteq v$ , then  $|v| < |u \cup v|$  which implies that

$$\begin{aligned} g(u) \wedge \tilde{b}_v &\leq g(u) \wedge (g(v) \wedge (g(u \cup v))^c) \\ &\leq g(u) \wedge (g(v) \wedge (g(u) \wedge g(v))^c) \\ &\leq 0, \end{aligned}$$

and if  $\tilde{b}_u \wedge \tilde{b}_v > 0$ , then  $g(u) \wedge \tilde{b}_v > 0$  and  $\tilde{b}_u \wedge g(v) > 0$ , i.e.  $u = v$ .

Also,  $\vee \{\tilde{b}_v : u \subseteq v\} = g(u)$  :

Suppose that  $0 < b = g(u) \wedge (\vee \{\tilde{b}_v : u \subseteq v\})^c$ , and  $\hat{b}_n = \vee \{g(v) : |v| = n\}$ . The sequence  $\langle \hat{b}_n : n \in \omega \rangle$  is decreasing, and  $\bigwedge \{\hat{b}_n : n \in \omega\} \leq \bigwedge \{b_n : n \in \omega\} = 0$  and  $\hat{b}_{|u|} \geq g(u) \geq b$ . Hence there exists  $k \in \omega$  such that  $b \wedge (\hat{b}_k)^c < b \wedge (\hat{b}_{k+1})^c$  i.e.  $(b \wedge \hat{b}_k) \wedge (\hat{b}_{k+1})^c > 0$ . But then there exists  $v \in \mathcal{P}_\omega(\lambda)$  such that  $|v| = k$  and  $(b \wedge g(v)) \wedge (\hat{b}_{k+1})^c > 0$ . Now  $(b \wedge g(v)) \wedge (\hat{b}_{k+1})^c = b \wedge (g(v) \wedge (\hat{b}_{k+1})^c)$  and  $g(v) \wedge (\hat{b}_{k+1})^c = \tilde{b}_v$ . Thus,  $g(u) \wedge \tilde{b}_v \geq b \wedge \tilde{b}_v > 0$  which implies that  $u \subseteq v$ , which is clearly impossible by the definition of  $b$ .

For each  $u \in \mathcal{P}_\omega(\lambda)$ , there exists  $f_u \in M[B]$  by Theorem 0.4.6 such that

$$\|\exists x \wedge \{\varphi_\alpha(x) : \alpha \in u\}\| = \|\bigwedge \{\varphi_\alpha(f_u) : \alpha \in u\}\|.$$

By Theorem 0.4.4, there exists  $h \in M[B]$  such that  $\|h = f_u\| \geq \tilde{b}_u$  for each  $u \in \mathcal{P}_\omega(\lambda)$ . If  $u \subseteq v$ , then

$$\begin{aligned} \|\bigwedge \{\varphi_\alpha(h) : \alpha \in u\}\| &\geq \|\bigwedge \{\varphi_\alpha(f_v) : \alpha \in u\}\| \wedge \|h = f_v\| \\ &\geq \|\bigwedge \{\varphi_\alpha(f_v) : \alpha \in v\}\| \wedge \|h = f_v\| \\ &\geq \tilde{b}_v. \end{aligned}$$

Thus,

$$\begin{aligned} \|\wedge \{\varphi_\alpha(h) : \alpha \in u\}\| &\geq \vee \{\tilde{b}_v : u \subseteq v\} \\ &\geq g(u) \in F, \end{aligned}$$

and the result follows.  $\square$

The above result emphasises the usefulness of good ultrafilters on Boolean algebras. Mansfield [1971] now focuses on the construction of such an ultrafilter and a saturated Boolean ultrapower.

A function  $f$  into  $B - \{0\}$  is *disjoint* if  $f(u) \wedge f(v) > 0$  implies that  $u = v$ .

**Definition 4.1.3 [R. Mansfield 1971] :**

A Boolean algebra  $B$  is  $\kappa$ -*disjointable* if for every function  $f$  into  $B - \{0\}$  with  $|f| < \kappa$ , there exists a disjoint function  $h$  into  $B - \{0\}$  such that  $h \leq f$ .

**Lemma 4.1.4 [R. Mansfield 1971] :**

*Suppose that  $B$  is  $\kappa$ -disjointable and  $D \subseteq B$  is closed under finite intersections with  $|D| < \kappa$ . If  $f : \mathcal{P}_\omega(\lambda) \rightarrow D$  is a monotonically decreasing function for  $\lambda < \kappa$ , then  $D$  can be extended to a set  $\tilde{D}$  such that  $\tilde{D}$  has the finite intersection property,  $|\tilde{D}| < \kappa$  and there exists a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow \tilde{D}$  with  $g \leq f$ .*

**Proof :**

Suppose that  $u \in \mathcal{P}_\omega(\lambda)$ ,  $d \in D$ . Then  $f(u) \wedge d > 0$ , and since  $B$  is  $\kappa$ -disjointable, there exists a disjoint function  $h(u, d)$  into  $B - \{0\}$  such that  $h(u, d) \leq f(u) \wedge d$ . Define a function  $g$  on  $\mathcal{P}_\omega(\lambda)$  by

$$g(u) = \vee \{h(v, d) : u \subseteq v \text{ and } d \in D\}.$$

If  $u \subseteq v$ , then

$$h(v, d) \leq f(v) \wedge d \leq f(u),$$

and hence  $g \leq f$ .

Also,  $g$  is multiplicative :

For  $u, \tilde{u} \in \mathcal{P}_\omega(\lambda)$ ,

$$\begin{aligned} g(u) \wedge g(\tilde{u}) &= \vee \{h(v, d) \wedge h(\tilde{v}, \tilde{d}) : u \subseteq v \text{ and } \tilde{u} \subseteq \tilde{v}\} \\ &= \vee \{h(v, d) : u, \tilde{u} \subseteq v\} \text{ since } h \text{ is disjoint} \\ &= \vee \{h(v, d) : u \cup \tilde{u} \subseteq v\} \\ &= g(u \cup \tilde{u}). \end{aligned}$$

Define

$$\tilde{D} = \{g(u) \wedge d : u \in \mathcal{P}_\omega(\lambda) \text{ and } d \in D\}.$$

$0 \notin \tilde{D}$ , since  $g(u) \wedge d \geq h(u, d) > 0$ . □

The above lemma suggests that a key to the existence of a  $\kappa$ -good ultrafilter is the existence of a  $\kappa$ -disjointable Boolean algebra.

Now suppose that  $\kappa$  is an inaccessible cardinal and  $B = RO(\pi\{\alpha : \alpha < \kappa\})$ , the regular open algebra of the product space  $\Pi\{\alpha : \alpha < \kappa\}$  with each factor having the discrete topology.

**Lemma 4.1.5 [R. Mansfield 1971] :**

*B is  $\kappa$ -disjointable.*

**Proof :**

Each open set in the space  $\Pi\{\alpha : \alpha < \kappa\}$  is an arbitrary union of base sets of the form  $\{\langle x_\alpha : \alpha < \kappa \rangle : x_{\alpha_i} \in A_i, 1 \leq i \leq n\}$  where  $A_i \subseteq \alpha_i$  for each  $1 \leq i \leq n$ . Hence the projection of an open set onto the  $\alpha$ th coordinate is all of  $\alpha$  for all but finitely many  $\alpha$ . Suppose that  $Y$  is a set of regular open sets with  $|Y| < \kappa$ . Then there exists  $\lambda < \kappa$  such that  $|Y| < \lambda$  and for each  $y \in Y$ , the projection of  $y$  onto the  $\lambda$ th coordinate is all of  $\lambda$ . Hence  $\lambda$  may be decomposed into  $|Y|$  non-empty, disjoint sets, say  $\lambda = \cup\{A_y : y \in Y\}$  where  $A_y \cap A_{\tilde{y}} = \emptyset$  if  $y \neq \tilde{y}$ . For any function  $f_\lambda$  into  $\Pi\{\alpha : \alpha < \kappa\} - \{0\}$ , the set  $\{y \cap f_\lambda(A_y) : y \in Y\}$  is a partition of  $Y$ . □

**Lemma 4.1.6 [R. Mansfield 1971] :**

*B satisfies the  $\kappa$ -chain condition.*

**Proof :**

Suppose that  $X$  is a set of non-empty, disjoint base sets with  $|X| \geq \kappa$ . Each base set can be identified with a finite function  $x$  which is such that  $x(\alpha) \subseteq \alpha$  for each  $\alpha < \kappa$ , and if  $\tilde{x}$  is any other such function with  $x(\alpha) \cap \tilde{x}(\alpha) = \emptyset$  for some

$\alpha$ , then the base sets which they have been identified with are disjoint. Suppose that  $X_n = \{x \in X : |x| = n\}$ . Then there exists  $n \in \omega$  such that  $|X_n| \geq \kappa$ . Now  $|X_0| < \kappa$ . Suppose that  $|X_k| < \kappa$ . If  $|X_{k+1}| \geq \kappa$ , then there exists  $\langle \alpha, A \rangle$  such that  $|\{x \in X_{k+1} : \alpha \in \text{dom } x \text{ and } x(\alpha) = A\}| \geq \kappa$ , which implies that  $|\{x : x \cup \{\langle \alpha, A \rangle\} \in X_{k+1}\}| \geq \kappa$ , a contradiction.  $\square$

**Lemma 4.1.7 [R. Mansfield 1971] :**

$$|B| = \kappa.$$

**Proof :**

Suppose that  $X$  is the set of all those elements of  $B$  which can be expressed as the join of fewer than  $\kappa$  base sets. Suppose that  $A$  is the subalgebra of  $B$  generated by  $X$ . Since  $\kappa$  is inaccessible, there are only  $\kappa$  base sets and  $|A| = \kappa$ . Now each element of  $B$  is the join of a disjoint subset of  $A$ . By Lemma 4.1.6,  $B$  satisfies the  $\kappa$ -chain condition. Thus  $|B| \leq |\{Y \subseteq \kappa : |Y| < \kappa\}| = \kappa$ .  $\square$

**Theorem 4.1.8 [R. Mansfield 1971] :**

*There exists a  $\kappa$ -good ultrafilter  $F$  on  $B$ .*

**Proof :**

For  $\lambda < \kappa$ ,  $|B^{\mathcal{P}_\omega(\lambda)}| = \kappa^\lambda = \kappa$ . Hence  $|\bigcup \{B^{\mathcal{P}_\omega(\lambda)} : \lambda < \kappa\}| = \kappa$ . The result follows by Theorem 4.1.4.  $\square$

**Corollary 4.1.9 [R. Mansfield 1971] :**

*If  $\aleph_0 \leq |M| < \kappa$ , then  $|M[B]/F| = \kappa$ .*

**Corollary 4.1.10 [R. Mansfield 1971] :**

*If  $|M| < \kappa$ , then  $M[B]/F$  is  $\kappa$ -saturated.*

## 4.2 The notion of goodness of an ultrafilter as defined by M. Benda [1974] and his complete generalisation of Keisler's theorem

Recall Benda's [1974] definition of a Boolean power from Chapter 0 (Section 0.2.3) :

Suppose that  $M$  is a model and  $B$  is a complete Boolean algebra. Suppose that  $P \in \mathbb{P}_B$ . The universe of the Boolean power  $M[B]$  is defined by

$$\{f \in M^P : P \in \mathbb{P}_B\}.$$



For  $f_1, \dots, f_n \in M[B]$  with domains  $P_1, \dots, P_n$  respectively, a relation  $R$  on  $M$  can be extended to a relation on  $M[B]$  by defining

$$R(f_1, \dots, f_n) = \vee \{p_1 \wedge \dots \wedge p_n : p_i \in P_i \text{ for } 1 \leq i \leq n \text{ and } M \models R(f_1(p_1), \dots, f_n(p_n))\}.$$

The Boolean ultrapower  $M[B]/F$  is then defined in the usual way.

**Definition 4.2.1 [M. Benda 1974] :**

An ultrafilter  $F$  on a Boolean algebra  $B$  is  $\kappa$ -good iff for each  $\lambda < \kappa$  and for each function  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$  for which there exists a family of partitions  $\{P_\alpha \in \mathbb{P}_B : \alpha < \lambda\}$  such that for each  $u \in \mathcal{P}_\omega(\lambda)$ ,  $f(u)$  is the join of elements of the coarsest common refinement of the  $P_\alpha, \alpha \in u$ , there exist a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  with  $g \leq f$ , and a partition  $P$  such that  $g(u)$  is the join of elements of the coarsest common refinement of  $P$  and the  $P_\alpha, \alpha \in u$ , and for each  $p \in P$ ,  $|\{\alpha : g(\{\alpha\}) \wedge p > 0\}| < \aleph_0$ .

**Remarks 4.2.2 :**

If  $B = \mathcal{P}(I)$  and  $F$  is  $\kappa$ -good by the above definition, then  $F$  is  $\kappa$ -good in the conventional sense : If  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$  is a monotonically decreasing function, then for each  $u \in \mathcal{P}_\omega(\lambda)$ ,  $f(u)$  is the join of elements of  $\{\{i\} : i \in I\}$ . Hence there exists a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  with  $g \leq f$ , which implies that the above definition is not weaker than the conventional one. Conversely, if  $B = \mathcal{P}(I)$  and  $F$  is  $\kappa$ -good in the conventional sense, then  $F$  is  $\kappa$ -good by the above definition : If  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$  is a function, then there exists a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  with  $g \leq f$ . Now for each  $u \in \mathcal{P}_\omega(\lambda)$ ,  $g(u)$  is the join of elements of  $\{\{i\} : i \in I\}$ , and  $|\{\alpha : g(\{\alpha\}) \cap \{i\} \neq \emptyset\}| < \aleph_0$  for each  $i \in I$ , since  $|\{\alpha : i \in g(\{\alpha\})\}| < \aleph_0$  for each  $i \in I$  by the multiplicativity of  $g$ . In general, if  $B$  is atomic, then the notion of  $\kappa$ -goodness as defined above coincides with the conventional notion, and is weaker than the conventional notion if  $B$  is atomless.

Benda requires the following lemma to obtain a complete generalisation of Keisler's theorem.

**Lemma 4.2.3 [M. Benda 1974] :**

*Suppose that  $\{A_\alpha : \alpha < \kappa\}$  is a disjoint family of non-empty sets. Suppose that  $f$  is a function on  $\mathcal{P}_\omega(\kappa)$  such that  $f(u) \subseteq \Pi\{A_\alpha : \alpha \in u\}$ , and if  $v \subseteq u$  and  $\langle a_1, \dots, a_k \rangle \in f(u)$ , then  $\langle a_i : i \in v \rangle \in f(v)$ . Then there exists a family of functions  $\{h_\alpha : \alpha < \kappa\}$  such that*

- (a)  $\text{dom } h_\alpha = A_\alpha$  for each  $\alpha < \kappa$ .
- (b) If  $u = \{\alpha_1, \dots, \alpha_n\} \in \mathcal{P}_\omega(\kappa)$  and  $\langle a_1, \dots, a_n \rangle \in \Pi\{A_\alpha : \alpha \in u\}$ , then  $\langle a_1, \dots, a_n \rangle \in f(u)$  iff  $\bigwedge \{h_{\alpha_i}(a_i) : 1 \leq i \leq n\} \neq 0$ .
- (c) If  $w \subset \kappa$  with  $|w| \geq \aleph_0$ , then  $\bigwedge \{h_\alpha(a_\alpha) : \alpha \in w\} = 0$ .

**Proof :**

For  $u \in \mathcal{P}_\omega(\lambda)$ , define by induction the function  $h^n$  on  $\cup\{\Pi\{A_\alpha : \alpha \in u\} : |u| = n\}$  such that

$$\{h^n(\langle a_1, \dots, a_n \rangle) : n \in \omega \text{ and } a_i \in A_{\alpha_i}\}$$

is pairwise disjoint, and if  $u = \{\alpha_1, \dots, \alpha_n\}$  and  $a_i \in A_{\alpha_i}$ , then

$$h^n(\langle a_1, \dots, a_n \rangle) \neq 0 \text{ iff } \langle a_1, \dots, a_n \rangle \in f(u).$$

For  $a \in A_\alpha$ , define

$$h_\alpha(a) = \vee \{h^n(\langle a_1, \dots, a_n \rangle) : n \in \omega \text{ and } a \in \{a_1, \dots, a_n\}\}.$$

(a) is true by the definition of  $h_\alpha$ . Suppose that  $u = \{\alpha_1, \dots, \alpha_n\}$ , and  $\langle a_1, \dots, a_n \rangle \in \Pi\{A_\alpha : \alpha \in u\}$ . If  $\langle a_1, \dots, a_n \rangle \in f(u)$  then  $h^n(\langle a_1, \dots, a_n \rangle) \neq 0$ , and  $h^n(\langle a_1, \dots, a_n \rangle) \leq h_{\alpha_i}(a_i)$  for each  $1 \leq i \leq n$ . Hence  $\wedge\{h_{\alpha_i}(a_i) : 1 \leq i \leq n\} \neq 0$ . Conversely, suppose that  $\wedge\{h_{\alpha_i}(a_i) : 1 \leq i \leq n\} \neq 0$ . Then there exist  $v \in \mathcal{P}_\omega(\kappa)$ ,  $v = \{\beta_1, \dots, \beta_k\}$ , and  $b_i \in A_{\beta_i}$  with  $\langle b_1, \dots, b_k \rangle \in f(v)$ , such that  $h^n(\langle b_1, \dots, b_k \rangle) \neq 0$  and  $h^n(\langle b_1, \dots, b_k \rangle) \leq \wedge\{h_{\alpha_i}(a_i) : 1 \leq i \leq n\}$ . Now by definition of  $h_\alpha$ , for each  $1 \leq i \leq n$ ,  $a_i \in \{b_1, \dots, b_k\}$ , and  $u \subseteq v$ . Then  $\langle a_i : i \in u \rangle = \langle b_j : j \in u \rangle \in f(u)$ , which proves (b). In order to prove (c), suppose that  $w \subset \kappa$ ,  $|w| \geq \aleph_0$ , and  $a_\alpha \in A_\alpha$  for each  $\alpha \in w$ . If  $\wedge\{h_\alpha(a_\alpha) : \alpha \in w\} \neq 0$ , then there exist  $u \in \mathcal{P}_\omega(\lambda)$ ,  $u = \{\alpha_1, \dots, \alpha_k\}$ , and  $b_i \in A_{\alpha_i}$  such that  $h^n(\langle b_1, \dots, b_k \rangle) \neq 0$  and  $h^n(\langle b_1, \dots, b_k \rangle) \leq \wedge\{h_\alpha(a_\alpha) : \alpha \in w\}$ , which implies that  $a_\alpha \in \{b_1, \dots, b_k\}$  for all  $\alpha \in w$ .  $\square$

**Theorem 4.2.4 [M. Benda 1974] :**

$M[B]/F$  is  $\kappa$ -saturated iff  $F$  is  $\kappa$ -good.

**Proof :**

Suppose that  $F$  is a  $\kappa$ -good ultrafilter on  $B$ . Suppose that  $\lambda < \kappa$ ,  $\bar{f}_\alpha$  is a finite sequence of elements of  $M[B]$  and  $\{\varphi_\alpha(x, \bar{f}_\alpha) : \alpha < \lambda\}$  is finitely satisfiable. For  $u \in \mathcal{P}_\omega(\lambda)$ , define a function  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$  by

$$f(u) = \|\exists x \wedge \{\varphi_\alpha(x, \bar{f}_\alpha) : \alpha \in u\}\|.$$

Suppose that  $P_\alpha$  is the coarsest common refinement of the domains of the elements of the sequence  $\bar{f}_\alpha$ . By definition of the Boolean value,  $f(\{\alpha\})$  is the join of elements of  $P_\alpha$ . Similarly, for each  $u \in \mathcal{P}_\omega(\lambda)$ ,  $f(u)$  is the join of elements of the coarsest common refinement of the  $P_\alpha$ ,  $\alpha \in u$ . Since  $F$  is  $\kappa$ -good,

there exist a multiplicative function  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  with  $g \leq f$ , and a partition  $P$ , such that for each  $p \in P$ ,  $|\{\alpha : g(\{\alpha\}) \wedge p > 0\}| < \aleph_0$ . Suppose that  $u_p = \{\alpha : g(\{\alpha\}) \wedge p > 0\}$  and  $X_p = \{x : x = a \wedge p > 0, \text{ for } a \text{ an element of the coarsest common refinement of the } P_\alpha, \alpha \in u_p\}$ .

Now  $X_p$  is disjoint :

If  $x, \tilde{x} \in X_p$ , then

$$\begin{aligned} x \wedge \tilde{x} &= a \wedge p \wedge \tilde{a} \wedge p, \text{ } a, \tilde{a} \text{ are elements of the} \\ &\text{coarsest common refinement of the } P_\alpha, \alpha \in u_p \\ &= 0. \end{aligned}$$

Also,  $\bigvee X_p = p$ . Hence  $X = \bigcup \{X_p : p \in P\}$  is a partition of  $B$ . For each  $x \in X$ , define

$$v_x = \{\alpha : x \leq g(\{\alpha\})\}.$$

If  $x \leq p$ , then  $v_x \subseteq u_p$  : If  $\alpha \in v_x$ , then  $0 < x \leq g(\{\alpha\})$  and since  $x \leq p$ ,  $0 < x \leq g(\{\alpha\}) \wedge p$  i.e.  $\alpha \in u_p$ .

Since  $g$  is multiplicative,  $g(\{\alpha\}) \leq g(v_x)$  for each  $\alpha \in v_x$ , and since  $x \leq g(\{\alpha\})$  for each  $\alpha \in v_x$ , it follows that  $x \leq g(v_x)$  for each  $x \in X$ . Also,  $g(v_x) \leq f(v_x)$ , and hence

$$x \leq f(v_x) = \|\exists y \wedge \{\varphi_\alpha(y, \bar{f}_\alpha) : \alpha \in v_x\}\|.$$

By Theorem 0.4.2, there exists a finite sequence  $\bar{d}_\alpha$  of elements of the domains of  $\bar{f}_\alpha$ , such that  $x \leq \bigwedge \bar{d}_\alpha$  and

$$M \models \exists y \wedge \{\varphi_\alpha(y, \bar{f}_\alpha(\bar{d}_\alpha)) : \alpha \in v_x\}.$$

Suppose that  $h \in M[B]$  such that

$$M \models \bigwedge \{\varphi_\alpha(h(x), \bar{f}_\alpha(\bar{d}_\alpha)) : \alpha \in v_x\}.$$

Since  $x \leq \bigwedge \bar{d}_\alpha$  for each  $\alpha \in v_x$ ,

$$\begin{aligned} \|\varphi_\alpha(h, \bar{f}_\alpha)\| &\geq x \\ &\geq \bigvee \{x : \alpha \in v_x\} = g(\{\alpha\}) \in F, \end{aligned}$$

and the implication from right to left follows.

In order to prove the reverse implication, suppose that  $F$  is an ultrafilter on  $B$  such that  $M[B]/F$  is  $\kappa$ -saturated for each model  $M$ . Suppose that  $f : \mathcal{P}_\omega(\lambda) \rightarrow F$  is a monotonically decreasing function, and  $\{P_\alpha : \alpha < \lambda\}$  is a family of

partitions of  $B$  such that  $f(u)$  is the join of elements of the coarsest common refinement of the  $P_\alpha, \alpha < \lambda$ .

By Lemma 4.2.3, there exist functions  $h_\alpha$  on  $P_\alpha$  such that

- (a) If  $u = \{\alpha_1, \dots, \alpha_n\} \in \mathcal{P}_\omega(\lambda)$  and  $a_i \in P_{\alpha_i}$  for  $1 \leq i \leq n$ , then  $\bigwedge \{a_i : 1 \leq i \leq n\} \neq 0$ , and  $\bigwedge \{a_i : 1 \leq i \leq n\} \leq f(u)$  iff  $\bigwedge \{h_{\alpha_i}(a_i) : 1 \leq i \leq n\} \neq 0$ .
- (b) If  $w \subset \kappa$  with  $|w| \geq \aleph_0$ , then  $\bigwedge \{h_\alpha(a_\alpha) : \alpha \in w\} = 0$ .

For each  $\alpha < \lambda$ ,  $h_\alpha$  may be chosen so that  $h_\alpha$  is a function into  $\mathcal{P}(I)$  for some set  $I$ . Suppose that  $M = \langle \mathcal{P}(I), \subseteq, \emptyset \rangle$ , and that  $\varphi_\alpha(x)$  is the formula  $x \subseteq h_\alpha/F$  and  $x \neq \emptyset$ . Clearly,  $\{\varphi_\alpha(x) : \alpha < \lambda\}$  is finitely satisfiable in  $M[B]/F$ . Since  $M[B]/F$  is  $\kappa$ -saturated, there exists a function  $h \in M[B]$  such that  $\|h \neq \emptyset\| \wedge \|h \subseteq h_\alpha\| \in F$  for each  $\alpha < \lambda$ . Suppose that  $P = \text{dom } h$  and

$$g(u) = \|h \neq \emptyset\| \wedge \bigwedge \{\|h \subseteq h_\alpha\| : \alpha \in u\}.$$

Then  $g$  is a function such that  $g : \mathcal{P}_\omega(\lambda) \rightarrow F$  and  $g$  is multiplicative. By Theorem 0.4.2,  $g(u)$  is the join of elements of the coarsest common refinement of  $P$  and the  $P_\alpha, \alpha \in u$ . If  $p \wedge \bigwedge \{p_\alpha : \alpha \in u\} \neq 0$ , and  $p \wedge \bigwedge \{p_\alpha : \alpha \in u\} \leq g(u)$  for  $p \in P$  and  $p_\alpha \in P_\alpha$ , then  $h(p) \neq \emptyset$  and  $h(p) \subseteq \bigcap \{h_\alpha(p_\alpha) : \alpha \in u\}$ . By (a),  $\bigwedge \{p_\alpha : \alpha \in u\} \leq f(u)$ , which implies that  $g \leq f$ . Finally, suppose that  $w \subset \kappa$  and  $|w| \geq \aleph_0$ . If  $\|h \neq \emptyset\| \wedge \|h \subseteq h_\alpha\| \wedge p \neq 0$  for each  $\alpha \in w$ , then  $h(p) \neq \emptyset$  and  $h(p) \subseteq \bigcap \{h_\alpha(p_\alpha) : \alpha \in w\}$ , contradicting (b). Hence  $|\{\alpha : g(\{\alpha\}) \wedge p > 0\}| < \aleph_0$ .  $\square$

---

## Chapter 5

### Limit Boolean ultrapowers

The notion of a limit ultrapower is a generalisation of the notion of an ordinary ultrapower. It has many of the useful properties of an ordinary ultrapower, for example, there exists a natural embedding from a model into any of its limit ultrapowers, and moreover, this embedding is elementary. Essentially, limit ultrapowers are special elementary submodels of ordinary ultrapowers. The term “limit ultrapower” could have been motivated by the fact that a limit ultrapower can be constructed as the direct limit of ordinary ultrapowers (see Chang and Keisler [1973]). Although the construction of a limit ultrapower is more complex than that of an ordinary ultrapower, limit ultrapowers facilitate the characterisation (up to isomorphism) of the complete extensions of a model.

K. Potthoff [1974] defines the notion of a limit Boolean ultrapower as a special submodel of a Boolean ultrapower of which the elements are in relation with an updirected set of regular subalgebras ordered by inclusion. With this definition, all the important results pertaining to limit ultrapowers become applicable to the Boolean case. Potthoff’s definition is motivated by his study of the full submodels of Boolean powers.

In this chapter a full account of Potthoff’s characterisation of the full submodels of Boolean powers will be given. His definition of a limit Boolean ultrapower, as well as the most important results which he obtained using this definition, will also be given. Potthoff uses Foster’s [1953] definition of a Boolean power to prove his results.

#### 5.1 The characterisation of the full submodels of a Boolean power of K. Potthoff [1974]

Recall (Definition 0.4.10) that a model  $M$  is full if for each formula  $\varphi$  and  $m_1, \dots, m_n \in M$ , there exists  $m \in M$  such that  $\|\exists x \varphi(x, m_1, \dots, m_n)\| = \|\varphi(m, m_1, \dots, m_n)\|$ . By Theorem 0.4.6, the Boolean power  $M[B]$  is full. In what follows, Potthoff [1974] addresses the problem of characterising those submodels of  $M[B]$  which are also full.

Recall the family of functions  $\{f^m : m \in M\}$  in the proof of Theorem 0.4.2 : For each  $m \in M$ , define  $f^m : M \rightarrow B$  by

$$f^m(x) = \begin{cases} 1, & x = m, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $f^m \in M[2]$ , and since  $M \cong M[2]$  by Theorem 0.3.1(b),  $M[2] = \{f^m : m \in M\}$  is an elementary submodel of  $M[B]$ . By Remark 0.4.3(b),  $\|f = f^m\| = f(m)$ , and since  $\vee\{\|f = f^m\| : m \in M\} = 1$ , by the application of equality axiom (iv), it follows that

$$\|\exists x \varphi(x, f_1, \dots, f_n)\| = \vee\{\|\varphi(f^m, f_1, \dots, f_n)\| : m \in M\}.$$

**Theorem 5.1.1 [K. Potthoff 1974] :**

*Suppose that  $M[2] \subseteq N \subseteq M[B]$ ,  $N$  is full, and  $F$  is an ultrafilter on  $B$ . Then  $N/F = \{f/F : f \in N\}$  is an elementary submodel of  $M[B]/F$ .*

**Proof :**

Suppose that  $\|\cdot\|_N$  denotes the Boolean value with respect to  $N$ . For  $f_1, \dots, f_n \in N$ ,  $\|\varphi(f_1, \dots, f_n)\|_N$  is identical to  $\|\varphi(f_1, \dots, f_n)\|$  if  $\varphi$  is an atomic formula, or a formula obtained from atomic formulas by applying the propositional connectives. Thus  $\|\cdot\|_N$  differs from  $\|\cdot\|$  only in the case of the existential quantifier.

Now

$$\begin{aligned} \|\exists x \varphi(x, f_1, \dots, f_n)\| &= \vee\{\|\varphi(f^m, f_1, \dots, f_n)\| : m \in M\} \text{ by a comment above} \\ &\leq \vee\{\|\varphi(f, f_1, \dots, f_n)\|_N : f \in N\} \text{ since } M[2] \subseteq N \\ &= \|\exists x \varphi(x, f_1, \dots, f_n)\|_N \\ &\leq \vee\{\|\varphi(f, f_1, \dots, f_n)\| : f \in M[B]\} \text{ since } N \subseteq M[B] \\ &= \|\exists x \varphi(x, f_1, \dots, f_n)\|. \end{aligned}$$

Hence  $\|\exists x \varphi(x, f_1, \dots, f_n)\|_N = \|\exists x \varphi(x, f_1, \dots, f_n)\|$ .

Also,  $N/F \models \varphi(f_1/F, \dots, f_n/F)$  iff  $\|\varphi(f_1, \dots, f_n)\|_N \in F$  :

If  $\varphi$  is atomic, then the result follows by definition.

If  $\varphi = \psi \vee \phi$ , then

$$\begin{aligned} \|\varphi(f_1, \dots, f_n)\|_N \in F &\text{ iff } \|\psi(f_1, \dots, f_n)\|_N \in F \text{ or } \|\phi(f_1, \dots, f_n)\|_N \in F \\ &\text{ iff } N/F \models \psi(f_1/F, \dots, f_n/F) \text{ or } N/F \models \phi(f_1/F, \dots, f_n/F) \\ &\text{ iff } N/F \models \psi(f_1/F, \dots, f_n/F) \vee \phi(f_1/F, \dots, f_n/F) \\ &\text{ iff } N/F \models \varphi(f_1/F, \dots, f_n/F). \end{aligned}$$

If  $\varphi = \ulcorner \psi$ , then

$$\begin{aligned} \|\varphi(f_1, \dots, f_n)\|_N \in F & \text{ iff } \|\psi(f_1, \dots, f_n)\|_N^c \in F \\ & \text{ iff } \|\psi(f_1, \dots, f_n)\|_N \notin F \\ & \text{ iff } N/F \not\models \psi(f_1/F, \dots, f_n/F) \\ & \text{ iff } N/F \models \varphi(f_1/F, \dots, f_n/F). \end{aligned}$$

If  $\varphi = \exists x \psi(x)$ , then since  $N$  is full,

$$\begin{aligned} \|\exists x \psi(x, f_1, \dots, f_n)\| \in F & \text{ iff } \exists f \in N [\|\psi(f, f_1, \dots, f_n)\|_N \in F] \\ & \text{ iff } \exists f \in N [N/F \models \psi(f/F, f_1/F, \dots, f_n/F)] \\ & \text{ iff } N/F \models \exists x \psi(x, f_1/F, \dots, f_n/F). \end{aligned}$$

Hence,

$$\begin{aligned} N/F \models \varphi(f_1/F, \dots, f_n/F) & \text{ iff } \|\varphi(f_1, \dots, f_n)\|_N \in F \\ & \text{ iff } \|\varphi(f_1, \dots, f_n)\| \in F \\ & \text{ iff } M[B]/F \models \varphi(f_1/F, \dots, f_n/F). \end{aligned}$$

□

**Corollary 5.1.2 [K. Potthoff 1974]:**

*Suppose that  $M[2] \subseteq N \subseteq M[B]$ ,  $N$  is full, and  $F$  is an ultrafilter on  $B$ . Then  $M$  is elementarily embeddable into  $N/F$ .*

**Remarks 5.1.3 :**

- (a) The subscript  $N$  in  $\|\cdot\|_N$  can be omitted if  $M[2] \subseteq N$ .
- (b) By replacing the formula  $\varphi(x)$  by the formula  $\exists x \varphi(x) \rightarrow \varphi(x)$ ,  $N$  is full iff for any formula  $\varphi(x)$  and  $f_1, \dots, f_n \in N$  such that  $\|\exists x \varphi(x, f_1, \dots, f_n)\| = 1$ , there exists  $f \in N$  such that  $\|\varphi(f, f_1, \dots, f_n)\| = 1$ .

The converse of Theorem 5.1.1 also holds.

**Theorem 5.1.4 [K. Potthoff 1974] :**

*Suppose that  $M[2] \subseteq N \subseteq M[B]$ , and  $F$  is an ultrafilter on  $B$ . If  $N/F$  is an elementary submodel of  $M[B]/F$ , then  $\cup N/F = \{f \in M[B] : \exists g \in N [\|f = g\| \in F]\}$  is full.*

**Proof :**

Suppose that  $f_1, \dots, f_n \in \cup N/F$  and  $\|\exists x\varphi(x, f_1, \dots, f_n)\| = 1$ . Then  $\|\exists x\varphi(x, f_1, \dots, f_n)\| \in F$  and hence  $M[B]/F \models \exists x\varphi(x, f_1/F, \dots, f_n/F)$ . Since  $N/F$  is an elementary submodel of  $M[B]/F$ ,  $N/F \models \exists x\varphi(x, f_1/F, \dots, f_n/F)$ , which implies that there exists  $f \in N$  such that  $N/F \models \varphi(f, f_1/F, \dots, f_n/F)$ . By applying the fact that  $N/F$  is an elementary submodel of  $M[B]/F$  once again, it follows that  $\|\varphi(f, f_1, \dots, f_n)\| \in F$ . By Remark 5.1.3(b) and the fact that  $M[B]$  is full, there exists  $g \in M[B]$  such that  $\|\varphi(g, f_1, \dots, f_n)\| = 1$ . Define a function  $h \in M[B]$  by

$$\|\varphi(f, f_1, \dots, f_n)\| \leq \|f = h\| \text{ and } \|\varphi(f, f_1, \dots, f_n)\|^c \leq \|g = h\|.$$

Then, since  $f \in N$  and  $\|f = h\| \in F$ ,  $h \in \cup N/F$ , and by equality axiom (iv),

$$\begin{aligned} \|\varphi(h, f_1, \dots, f_n)\| &\geq \|\varphi(f, f_1, \dots, f_n)\| \wedge \|f = h\| \\ &= \|\varphi(f, f_1, \dots, f_n)\|, \end{aligned}$$

as well as

$$\begin{aligned} \|\varphi(h, f_1, \dots, f_n)\| &\geq \|\varphi(g, f_1, \dots, f_n)\| \wedge \|g = h\| \\ &\geq \|g = h\| \\ &\geq \|\varphi(f, f_1, \dots, f_n)\|^c. \end{aligned}$$

Hence

$$\begin{aligned} \|\varphi(h, f_1, \dots, f_n)\| &\geq \|\varphi(f, f_1, \dots, f_n)\| \vee \|\varphi(f, f_1, \dots, f_n)\|^c \\ &= 1, \end{aligned}$$

and the result follows. □

Frayne in Frayne, Morel and Scott [1962] shows that any two models  $M$  and  $K$  are elementarily equivalent iff  $K$  is elementarily embeddable into an ultrapower of  $M$ . Potthoff [1974] devises the following Boolean version of Frayne's Lemma in order to prove his main result in this section.

**Lemma 5.1.5 [K. Potthoff 1974] :**

*Two models  $M$  and  $K$  are elementarily equivalent iff  $K$  is elementarily embeddable into a Boolean ultrapower  $M[B]/F$  of  $M$ . Suppose that the language of  $M$  has been expanded by adjoining all constants in  $M$  or in  $K$ , and suppose that  $\Gamma$  is the set of all the formulas of this expanded language. Then  $B$  can be chosen as*



the completion of  $\Gamma / \sim Th(M, \bar{m})$ , and  $F$  can be any ultrafilter which includes  $Th(K, \bar{k}) / \sim Th(M, \bar{m})$  ( $\bar{m}$  and  $\bar{k}$  are enumerations of  $M$  and  $K$ ).

**Proof :**

Suppose that  $K$  is elementarily embeddable into a Boolean ultrapower  $M[B]/F$ . Then since  $M$  is also elementarily embeddable into  $M[B]/F$  (Corollary 0.5.3),  $M$  and  $K$  are elementarily equivalent.

Conversely, suppose that  $M$  and  $K$  are elementarily equivalent. To simplify the proof, suppose that  $c_m$  is the individual constant interpreted as  $m$  in  $(M, \bar{m})$  and  $(K, \bar{k})$  for all elements in  $K \cup M$ , and that  $K \cap M = \emptyset$ . Suppose that  $B$  is the completion of  $\Gamma / \sim Th(M, \bar{m})$ . For any formula  $\varphi \in \Gamma$ , suppose that  $\varphi / \sim$  is the equivalence class of  $\varphi$  w.r.t.  $Th(M, \bar{m})$ , and that  $F$  is an ultrafilter on  $B$  such that  $\varphi / \sim \in F$  whenever  $\varphi \in Th(K, \bar{k})$ . By equality axiom (iv),

$$(c_{m_1} = c_{k_1}) / \sim \wedge \cdots \wedge (c_{m_n} = c_{k_n}) / \sim \leq \varphi(c_{k_1}, \dots, c_{k_n}) / \sim \text{ for } m_1, \dots, m_n \in M$$

such that  $M \models \varphi(m_1, \dots, m_n)$ .

Now suppose that

$$\begin{aligned} & \vee \{ (c_{m_1} = c_{k_1}) / \sim \wedge \cdots \wedge (c_{m_n} = c_{k_n}) / \sim : M \models \varphi(m_1, \dots, m_n) \} \\ & < \varphi(c_{k_1}, \dots, c_{k_n}) / \sim . \end{aligned}$$

Since  $\Gamma / \sim Th(K, \bar{k})$  is dense in  $B$ , there exists  $\psi \in \Gamma$  such that

$$\psi / \sim \leq (\vee \{ (c_{m_1} = c_{k_1}) / \sim \wedge \cdots \wedge (c_{m_n} = c_{k_n}) / \sim : M \models \varphi(m_1, \dots, m_n) \})^c \wedge \varphi(c_{k_1}, \dots, c_{k_n}) / \sim .$$

By the substitution axiom, it can be assumed w.l.o.g. that the only individual constants in  $\psi$  denoting elements of  $K$  are  $c_{k_1}, \dots, c_{k_n}$ . Now since  $(\varphi \wedge \neg \psi) / \sim \neq 0$ ,  $Th(M, \bar{m}) \cup \{\varphi, \neg \psi\}$  is consistent, and hence has a model  $(Q, \bar{q})$ . Hence

$$\exists x_1 \dots \exists x_n (\varphi(x_1, \dots, x_n) \wedge \neg \psi(c_{k_1} | x_1, \dots, c_{k_n} | x_n))$$

holds in  $(Q, \bar{q})$  and therefore in  $(M, \bar{m})$  i.e. there exists  $m_1, \dots, m_n \in M$  such that

$$\varphi(c_{m_1}, \dots, c_{m_n}) \wedge \neg \psi(c_{k_1} | c_{m_1}, \dots, c_{k_n} | c_{m_n}) \in Th(M, \bar{m}).$$

By assumption

$$((c_{m_1} = c_{k_1}) \wedge \cdots \wedge (c_{m_n} = c_{k_n}) \rightarrow \psi(c_{k_1}, \dots, c_{k_n})) / \sim = 1$$

$$\text{i.e. } (\lceil \psi(c_{k_1}, \dots, c_{k_n}) \rceil (c_{m_1} = c_{k_1}) \vee \cdots \vee \lceil (c_{m_n} = c_{k_n}) \rceil) / \sim = 1.$$

However, there exists a model of  $Th(M, \overline{m})$  and interpretations of  $c_{k_1}, \dots, c_{k_n}$  such that

$$(M, \overline{m} \cup \{k_1, \dots, k_n\}) \models \lceil \psi(c_{k_1}, \dots, c_{k_n}) \wedge (c_{m_1} = c_{k_1}) \wedge \cdots \wedge (c_{m_n} = c_{k_n}) \rceil,$$

a contradiction. Hence

$$\begin{aligned} & \vee \{(c_{m_1} = c_{k_1}) / \sim \wedge \cdots \wedge (c_{m_n} = c_{k_n}) / \sim : M \models \varphi(m_1, \dots, m_n)\} \\ & = \varphi(c_{k_1}, \dots, c_{k_n}) / \sim. \end{aligned}$$

Now suppose that  $\varphi$  is the formula  $x_1 = x_1$ . Then

$$\begin{aligned} \vee \{(c_m = c_k) / \sim : m \in M\} &= (c_k = c_k) / \sim \\ &= 1 \quad \text{for each } k \in K. \end{aligned}$$

Define a function  $e : K \rightarrow B^M$  by

$$e(k)(m) = (c_m = c_k) / \sim$$

$$\vee \{e(k)(m) : m \in M\} = \vee \{(c_m = c_k) / \sim : m \in M\} = 1, \text{ and for } m_1 \neq m_2,$$

$$\begin{aligned} e(k)(m_1) \wedge e(k)(m_2) &= (c_{m_1} = c_k) / \sim \wedge (c_{m_2} = c_k) / \sim \\ &\leq (c_{m_1} = c_{m_2}) / \sim \\ &= 0. \end{aligned}$$

Hence  $e : K \rightarrow M[B]$ .

Suppose that  $K \models \varphi(k_1, \dots, k_n)$ . Then  $\varphi(c_{k_1}, \dots, c_{k_n}) \in Th(K, \overline{k})$  and by definition of  $F$ ,  $\varphi(c_{k_1}, \dots, c_{k_n}) / \sim \in F$ . Since

$$\begin{aligned} \varphi(c_{k_1}, \dots, c_{k_n}) / \sim &= \vee \{(c_{m_1} = c_{k_1}) / \sim \wedge \cdots \wedge (c_{m_n} = c_{k_n}) / \sim : M \models \varphi(m_1, \dots, m_n)\} \\ &= \vee \{e(k_1)(m_1) \wedge \cdots \wedge e(k_n)(m_n) : M \models \varphi(m_1, \dots, m_n)\} \\ &= \|\varphi(e(k_1), \dots, e(k_n))\|, \end{aligned}$$

it follows that  $M[B]/F \models \varphi(e(k_1)/F, \dots, e(k_n)/F)$ , and hence  $K$  is elementarily embeddable into  $M[B]/F$ .  $\square$

Potthoff's [1974] main result now follows.

**Theorem 5.1.6 [K. Potthoff 1974] :**

*$M$  is elementarily embeddable into  $K$  iff there exist a complete Boolean algebra  $B$ , an ultrafilter  $F$  on  $B$  and a full model  $N$  with  $M[2] \subseteq N \subseteq M[B]$ , such that  $K \cong N/F$ .*

**Proof :**

Suppose that  $K \cong N/F$ . Then, since  $M$  is elementarily embeddable into  $N/F$  by Corollary 5.1.2,  $M$  is elementarily embeddable into  $K$ . Conversely, suppose that the function  $\tilde{e} : M \rightarrow K$  is an elementary embedding. Hence  $Th(M, \overline{m}) \cup \{(c_m = c_{\tilde{e}(m)}) : m \in M\}$  is consistent. Suppose that  $\Gamma$  and  $F$  are defined as in Lemma 5.1.5, and  $B$  is the completion of  $\Gamma / \sim Th(M, \overline{m}) \cup \{(c_m = c_{\tilde{e}(m)}) : m \in M\}$ . If the function  $e : K \rightarrow M[B]$  is defined as in Lemma 5.1.5, then by Lemma 5.1.5, the function from  $K$  into  $M[B]/F$  induced by  $e$  is an elementary embedding and maps  $\tilde{e}(m)$  onto  $f^m/F$ . Suppose that

$$N = \{f \in M[B] : \exists k \in K [ \|f = e(k)\| \in F ]\}.$$

Then  $M[2] \subseteq N$  and  $\cup N/F = N$ . The range of the elementary embedding from  $K$  into  $M[B]/F$  is  $N/F$ , and  $N/F$  is an elementary submodel of  $M[B]/F$ . By Theorem 5.1.4,  $\cup N/F = N$  is full.  $\square$

## 5.2 The definition of a limit Boolean ultrapower of K. Potthoff [1974] and his results concerning limit Boolean ultrapowers

Suppose that  $T$  is an updirected set of regular subalgebras of  $B$  partially ordered by inclusion.

**Definition 5.2.1 [K. Potthoff 1974] :**

The *limit Boolean ultrapower* is the submodel  $M[B] \upharpoonright T/F$  of  $M[B]/F$  where

$$M[B] \upharpoonright T = \{f \in M[B] : \exists C \in T [\text{range } f \subseteq C]\}.$$

Suppose that the language of  $M$  is expanded by adjoining a symbol for each of the relations and functions on  $M$ . The model  $\overline{M}$  in which every relation  $R$  and

every function  $f$  is the interpretation of a relation symbol  $\overline{R}$ , and a function symbol  $\overline{f}$  respectively, is called a *complete* model.  $\overline{M}$  is the completion of  $M$ .

The following theorem provides some insight into the relationship between full submodels of a Boolean power and limit Boolean ultrapowers.

**Theorem 5.2.2 [K. Potthoff 1974] :**

*Suppose that  $M[2] \subseteq N \subseteq M[B]$  and  $M$  is infinite. Then  $N$  is full w.r.t. the language  $\overline{L}$  of  $\overline{M}$  iff  $N = M[B] \upharpoonright T$  where  $T$  is an updirected family of regular subalgebras of  $B$ .*

**Proof :**

Suppose that  $N = M[B] \upharpoonright T$  and  $\varphi(x, x_1, \dots, x_n)$  is a formula of  $\overline{L}$ . For  $f_1, \dots, f_n \in N$ , there exist regular subalgebras  $C_1, \dots, C_n$  and  $C$  of  $B$  such that  $\text{range } f_i \subseteq C_i$  for each  $1 \leq i \leq n$  and  $\cup\{C_i : 1 \leq i \leq n\} \subseteq C$ . Since  $C$  is regular,  $M[2] \subseteq M[C] \subseteq N$ , and hence, by the same method as in the proof of Theorem 5.1.1, it follows that  $\|\psi(g_1, \dots, g_n)\|_{M[C]} = \|\psi(g_1, \dots, g_n)\| = \|\psi(g_1, \dots, g_n)\|_N$  for any formula  $\psi$  and  $g_1, \dots, g_n \in M[C]$ . Now since  $M[C]$  is full w.r.t.  $\overline{L}$ , there exists  $f \in M[C] \subseteq N$  such that  $\|\exists x \varphi(x, f_1, \dots, f_n)\|_{M[C]} = \|\varphi(f, f_1, \dots, f_n)\|_{M[C]}$ . Also,

$$\begin{aligned} \|\exists x \varphi(x, f_1, \dots, f_n)\|_N &= \|\exists x \varphi(x, f_1, \dots, f_n)\|_{M[C]} \\ &= \|\varphi(f, f_1, \dots, f_n)\|_{M[C]} \\ &= \|\varphi(f, f_1, \dots, f_n)\|_N, \end{aligned}$$

and hence  $N$  is full.

Conversely, suppose that  $M[2] \subseteq N \subseteq M[B]$ , and  $N$  is full. For each  $f \in N$ , suppose that  $C_f$  is the regular subalgebra of  $B$  generated by the range of  $f$ . Define

$$T = \{C_f : f \in N\}.$$

Suppose that  $h : M \rightarrow M \times M$  is a bijection, and  $h_1$  and  $h_2$  are functions on  $M$  such that  $h(m) = \langle h_1(m), h_2(m) \rangle$  for each  $m \in M$ . Since  $\overline{M}$  is complete, there exists a relation  $R$  on  $M$  such that  $R = h$ . Since  $h$  is a bijection,  $\overline{M} \models \forall x_1 \forall x_2 \exists! x \overline{R}(x, x_1, x_2)$ , and hence  $\|\forall x_1 \forall x_2 \exists! x \overline{R}(x, x_1, x_2)\| = 1$ . For any  $f_1, f_2 \in N$ ,  $\|\exists! x \overline{R}(x, f_1, f_2)\| = 1$ , and since  $N$  is full w.r.t.  $\overline{L}$ , there exists  $f \in N$  such that  $\|\overline{R}(f, f_1, f_2)\| = 1$ . Hence

$$\begin{aligned} 1 &= \vee \{f(m) \wedge f_1(m_1) \wedge f_2(m_2) : \overline{M} \models \overline{R}(m, m_1, m_2)\} \\ &= \vee \{f(m) \wedge f_1(h_1(m)) \wedge f_2(h_2(m)) : m \in M\}, \end{aligned}$$

and since  $h$  is injective,  $f(m) = f_1(h_1(m)) \wedge f_2(h_2(m))$  for each  $m \in M$ . Since  $h$  is surjective,  $\text{range } f_1 \cup \text{range } f_2 \subseteq C_f$ , and hence  $C_{f_1} \cup C_{f_2} \subseteq C_f$ , which implies that  $T$  is updirected by inclusion. It remains to be shown that  $N = M[B] \upharpoonright T$ . Suppose that  $g \in M[B] \upharpoonright T$ . Then  $\text{range } g \subseteq C_f$  for some  $f \in N$ . Fix  $\tilde{m} \in M$  and define a function  $e : M \rightarrow M$  by

$$e(m) = \begin{cases} \text{the unique } m_1, \text{ such that } f(m) \leq g(m_1) \text{ whenever } f(m) \neq 0, \\ \tilde{m}, & \text{otherwise.} \end{cases}$$

Since  $\overline{M}$  is complete, there exists a relation  $S$  on  $M$  such that  $S = e$ , and  $\overline{M} \models \forall x \exists! x_1 \overline{S}(x, x_1)$ . Hence  $\|\exists! x_1 \overline{S}(f, x_1)\| = 1$ , and since  $N$  is full w.r.t.  $\overline{L}$ , there exists  $f_1 \in N$  such that  $\|\overline{S}(f, f_1)\| = 1$ . Then

$$\begin{aligned} 1 &= \vee \{f(m) \wedge f_1(m_1) : \overline{M} \models \overline{S}(m, m_1)\} \\ &= \vee \{f(m) \wedge f_1(g(m_1)) : m \in M\}. \end{aligned}$$

Hence  $f_1(m_1) = \vee \{f(m) : e(m) = m_1\} = g(m_1)$  for all  $m_1 \in M$ , which implies that  $g \in N$ . Clearly, if  $g \in N$ , then  $g \in M[B] \upharpoonright T$  by the definition of  $T$ , and hence  $N = M[B] \upharpoonright T$ .  $\square$

**Remark 5.2.3 :**

The limit Boolean ultrapower  $M[B] \upharpoonright T/F$  is the direct limit of the family  $\{M[C_f]/F \cap C_f : f \in N\}$ . Since  $C_f$  is atomic for each  $f \in N$ ,  $M[C_f]/F \cap C_f$  is isomorphic to an ordinary ultrapower and hence the following result holds.

**Corollary 5.2.4 :**

*Any limit Boolean ultrapower is the direct limit of ordinary ultrapowers.*

Recall that S. Koppelberg and B. Koppelberg [1976] show that there exists a model  $\widetilde{M}$  such that the class of ultrapowers of  $\widetilde{M}$  is a proper subclass of the class of Boolean ultrapowers of  $\widetilde{M}$  (see Chapter 1). However, the same relation does not hold between the class of limit ultrapowers and the class of limit Boolean ultrapowers of any model  $M$ . S.B. Kochen in Chang and Keisler [1973] shows that any limit ultrapower is the direct limit of ordinary ultrapowers, and by applying Corollary 5.2.4, the following result holds.

**Theorem 5.2.5 :**

*Any limit Boolean ultrapower is isomorphic to a limit ultrapower.*

An immediate consequence of the above result is that the Boolean analogue of Kochen's Limit Ultrapower Theorem (see Bell and Slomson [1971]) holds.

**Definition 5.2.6 :**

Suppose that  $M$  and  $K$  are models, and  $\overline{M}$  is the completion of  $M$ . An embedding  $e : M \rightarrow K$  is a *strong embedding* if there exists an extension  $\tilde{K}$  of  $K$  such that  $e : \overline{M} \rightarrow \tilde{K}$  is elementary.  $M$  is *strongly embeddable* into  $K$ .

**Proposition 5.2.7 :**

Suppose that  $f^m$  is defined as in the proof of Theorem 0.4.2. Then the embedding  $e : M \rightarrow M[B]/F$  defined by

$$e(m) = f^m / F$$

is a strong embedding.

**Proposition 5.2.8 :**

Suppose that  $C$  is a regular subalgebra of  $B$ . Then the embedding  $e : M[C]/F \cap C \rightarrow M[B]/F$  defined by

$$e(f/F \cap C) = f/F$$

is a strong embedding.

The following result is the Boolean analogue of Corollary 6.4.11 in Chang and Keisler [1973] and, as in the case of limit ultrapowers, permits the characterisation (up to isomorphism) of the complete extensions of a model.

**Theorem 5.2.9 [K. Potthoff 1974] :**

$M$  is strongly embeddable into  $K$  iff  $K$  is isomorphic to a limit Boolean ultrapower of  $M$ .

**Proof :**

If  $M$  is strongly embeddable into  $K$ , then there exist an extension  $\tilde{K}$  of  $K$  and an embedding  $e : M \rightarrow K$  such that  $e : \overline{M} \rightarrow \tilde{K}$  is elementary. By Theorem 5.1.6, this is equivalent to the condition that there exist a complete Boolean algebra  $B$ , an ultrafilter  $F$  on  $B$ , and a model  $N$  which is full w.r.t.  $\overline{L}$  and which satisfies  $M[2] \subseteq N \subseteq M[B]$  such that  $\tilde{K} \cong N/F$ . By Theorem 5.2.2,  $N$  is full w.r.t.  $\overline{L}$  iff  $N = M[B] \upharpoonright T$  for a family  $T$  of regular subalgebras of  $B$ .  $\square$

**Remark 5.2.10 :**

Consider the model  $\langle \omega, < \rangle$ . If  $F$  is descendingly countably incomplete, then by Theorem 3.2,  $\langle \omega, < \rangle[B]/F$  is  $\aleph_1$ -saturated. On the other hand, if  $F$  is descendingly countably complete, then  $\langle \omega, < \rangle \cong \langle \omega, < \rangle[B]/F$ . However, there exist complete extensions of  $\langle \omega, < \rangle$  which are neither  $\aleph_1$ -saturated nor isomorphic to  $\langle \omega, < \rangle$ . Hence the class of Boolean ultrapowers of  $\langle \omega, < \rangle$  is a proper subclass of the class of limit Boolean ultrapowers of  $\langle \omega, < \rangle$ .

The above observation firmly establishes the notion of a limit Boolean ultrapower as a generalisation of the notion of a Boolean ultrapower. It confirms the fact that Corollary 5.2.4 (when compared to the construction of Ouwehand and Rose [1998] of a Boolean ultrapower) and the Boolean analogue of Kochen's Limit Ultrapower Theorem (when compared to Mansfield's [1971] version of the Keisler-Shelah theorem) are results in their own right since they pertain to a larger class of models.

---

## Chapter 6

### Simple and Subdirectly irreducible Boolean ultrapowers

It is plausible that the ultrapowers of a simple algebra are simple. However, Frayne, Morel and Scott [1962] give an example of an ultrapower of a simple group which is not simple. This implies that there exist Boolean ultrapowers of a simple algebra which are not simple, and S. Burris and E. Jeffers [1978] investigate necessary and sufficient conditions for a Boolean ultrapower to be simple, or subdirectly irreducible, provided that the language is countable.

In this chapter the results of Burris and Jeffers will be the focus of the discussion. They use Foster's original definition of a Boolean power to obtain their results.

An algebra  $M$  is *simple* if  $|M| > 1$  and the only congruences on  $M$  are  $\Delta_M$  and  $\nabla_M$ , where  $\Delta_M = \{\langle m, m \rangle : m \in M\}$  and  $\nabla_M = M \times M$ .  $M$  is  $\langle m_1, m_2 \rangle$ -*irreducible* if  $m_1 \neq m_2$  and each non-trivial congruence on  $M$  identifies  $m_1$  and  $m_2$ .  $M$  is *subdirectly irreducible* if there exists  $m_1, m_2 \in M$  such that  $M$  is  $\langle m_1, m_2 \rangle$ -irreducible.

**Definition 6.1** [S. Burris and E. Jeffers [1978]] :

A first-order sentence  $\varphi$  is a *simplicity sentence* if all models of  $\varphi$  are simple. A *subdirect irreducibility sentence* is defined similarly. Burris and Jeffers [1978] require the following result of W. Taylor [1972].

**Proposition 6.2** [W. Taylor 1972] :

Suppose that  $M$  is an algebra and  $\theta$  is a congruence on  $M$  generated by  $\langle m_1, m_2 \rangle$ . Then for any  $\langle m_3, m_4 \rangle \in M \times M$ ,  $\langle m_3, m_4 \rangle \in \theta$  iff there exists a first-order formula  $\varphi(x_1, x_2, x_3, x_4)$  in the language of  $M$  such that :

- (a)  $\varphi$  is positive,
- (b)  $\vdash \forall m_3 \forall m_4 \exists x \varphi(x, x, m_3, m_4) \rightarrow m_3 = m_4$ ,
- (c)  $M \models \varphi(m_1, m_2, m_3, m_4)$ .

For the rest of the discussion, assume that the language of  $M$  is countable.

**Lemma 6.3** [S. Burris and E. Jeffers 1978] :

Suppose that  $M$  is an  $\aleph_0$ -saturated algebra. Then  $M$  satisfies a simplicity sentence (subdirect irreducibility) sentence iff  $M$  is simple (subdirectly irreducible).



**Proof :**

Trivially, if  $M$  satisfies a simplicity sentence, then  $M$  is simple. Conversely, suppose that  $M$  does not satisfy a simplicity sentence. By Proposition 6.2,  $\langle m_3, m_4 \rangle \in \theta(\langle m_1, m_2 \rangle)$  iff there exists a first-order formula  $\varphi(x_1, x_2, x_3, x_4)$  satisfying (a), (b) and (c). Suppose that  $\{\varphi_n(x_1, x_2, x_3, x_4) : n \in \omega\}$  is the set of all such formulas in the language of  $M$ . From Proposition 6.2 it follows that:

(a)  $M$  is not simple iff there exist  $m_1, m_2, m_3, m_4 \in M$ , such that

$$m_1 \neq m_2 \text{ and } M \models \bigwedge \varphi_n(m_1, m_2, m_3, m_4) \text{ for all } n \in \omega,$$

(b) For each finite subset  $\{\varphi_{n_1}(x_1, x_2, x_3, x_4), \dots, \varphi_{n_k}(x_1, x_2, x_3, x_4)\}$ ,

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 (x_1 \neq x_2 \rightarrow \varphi_{n_1}(x_1, x_2, x_3, x_4) \vee \dots \vee \varphi_{n_k}(x_1, x_2, x_3, x_4))$$

is a simplicity sentence.

Since  $M$  does not satisfy any simplicity sentence, for each  $n_1, \dots, n_k \in \omega$ ,

$$M \models \bigwedge \varphi_{n_1}(x_1, x_2, x_3, x_4) \vee \dots \vee \varphi_{n_k}(x_1, x_2, x_3, x_4).$$

Suppose that  $\Gamma$  is the set of all formulas of the form

$$x_1 \neq x_2 \text{ and } \bigwedge \varphi_n(x_1, x_2, x_3, x_4).$$

Then  $\Gamma$  is finitely satisfiable in  $M$ . Since  $M$  is  $\aleph_0$ -saturated and the formulas in  $\Gamma$  contain no parameters from  $M$ ,  $\Gamma$  is satisfiable in  $M$ . By (a),  $M$  is not simple. A similar argument holds in the case of subdirect irreducibility.  $\square$

The above lemma enables Burris and Jeffers [1978] to establish the following necessary and sufficient conditions for a Boolean ultrapower to be simple (subdirectly irreducible).

**Theorem 6.4 [S. Burris and E. Jeffers 1978] :**

*Suppose that  $F$  is an ultrafilter on a complete Boolean algebra  $B$ . Then  $M[B]/F$  is simple (subdirectly irreducible) iff  $M$  is simple (subdirectly irreducible) and  $F$  is descendingly countably complete, or  $M$  satisfies a simplicity (subdirect irreducibility) sentence.*

**Proof :**

If  $M$  satisfies a simplicity sentence, then  $M$  is simple, and since  $M$  is elementarily embeddable into  $M[B]/F$ ,  $M[B]/F$  is simple. Conversely, suppose that  $M$  is simple and  $F$  is descendingly countably complete. Suppose that  $\langle \varphi_n : n \in \omega \rangle$

is an enumeration of the formulas in the proof of Lemma 6.3. Suppose that  $f_1, f_2, f_3, f_4 \in M[B]$  such that  $f_1/F \neq f_2/F$ . Then, since  $M$  is simple,

$$\forall \{ \|\varphi_n(f_1, f_2, f_3, f_4)\| : n \in \omega \} \in F,$$

and since  $F$  is descendingly countably complete, there exists  $n \in \omega$  such that

$$\|\varphi_n(f_1, f_2, f_3, f_4)\| \in F$$

i.e.  $M[B]/F \models \varphi_n(f_1/F, f_2/F, f_3/F, f_4/F)$ , and hence  $M[B]/F$  is simple.

If  $M[B]/F$  is simple, but  $F$  is descendingly countably incomplete, then by Theorem 3.2,  $M[B]/F$  is  $\aleph_1$ -saturated and hence  $\aleph_0$ -saturated. By Lemma 6.3,  $M[B]/F$  satisfies a simplicity sentence, and since  $M \equiv M[B]/F$ , the result follows. A similar argument holds in the case of subdirect irreducibility.  $\square$

**Corollary 6.5 [S. Burris and E. Jeffers 1978] :**

*$M[B]/F$  is simple (subdirectly irreducible) iff  $M$  satisfies a simplicity (subdirect irreducibility) sentence.*

In view of the fact that the notion of a Boolean ultrapower is a generalisation of the notion of an ordinary ultrapower, Theorem 6.4 also holds for ultrapowers, and more generally, for ultraproducts.

**Problem 6.6 :**

Find necessary and sufficient conditions for the  $(B, P)$ -ultraproduct  $\Pi\{M_p[B \upharpoonright p] : p \in P\}/F$  as defined in Section 0.6 to be simple (subdirectly irreducible).

---

## Chapter 7

### Rudin-Keisler posets of complete Boolean algebras

From Chapter 1 (Definition 1.1.4), recall that for any two ultrafilters  $F$  and  $G$  on  $\mathcal{P}(I)$  and  $\mathcal{P}(J)$  respectively,  $F \leq G$  in the Rudin-Keisler ordering if there exists a function  $f : J \rightarrow I$  such that for each  $X \subseteq I$ ,  $X \in F$  iff  $f^{-1}[X] \in G$ . A. Blass [1970] characterises the Rudin-Keisler ordering on ultrafilters on powerset Boolean algebras in terms of elementary embeddings between ordinary ultrapowers. In order to obtain a Boolean version of Blass' Characterisation Theorem, it is necessary to extend the Rudin-Keisler ordering to ultrafilters on complete but not necessarily atomic Boolean algebras, since in the construction of a Boolean ultrapower, an arbitrary complete Boolean algebra replaces the usual  $\mathcal{P}(I)$ . P. Jipsen, A. Pinus and H. Rose [2000] propose such an extension which could possibly have been inspired by the duality between the category of sets with functions and the category of powerset Boolean algebras with complete homomorphisms. The category they consider as dual to the category of complete Boolean algebras with complete homomorphisms is the category of lattices of partitions of these Boolean algebras with concurrent families of functions.

In this chapter the extension of the Rudin-Keisler ordering to ultrafilters on complete Boolean algebras as given by Jipsen et al [2000] will be discussed. Their Boolean version of Blass' [1970] Characterisation Theorem, as well as the conditions for the Rudin-Keisler posets of a Boolean algebra to be embeddable into the Rudin-Keisler posets of another Boolean algebra will also be discussed. They use the definition of the Boolean power as given by Ouwehand and Rose [1998] to obtain their results.

#### 7.1 The extension of the Rudin-Keisler ordering to ultrafilters on complete Boolean algebras of P. Jipsen et al [2000]

For the sake of clarity, some of the details of the approach of Ouwehand and Rose [1998] (see Chapter 0) are briefly recalled below.

Suppose that  $B$  is a complete Boolean algebra and  $\mathbb{P}_B$  is the set of all partitions of  $B$ . The refinement relation  $\leq_r$  between partitions is a partial order on  $\mathbb{P}_B$ . Note that the ordering  $\leq_r$  on  $\mathbb{P}_B$  is reverse to the ordering  $\leq$  imposed on  $\mathbb{P}_B$  in the approach of Ouwehand and Rose [1998]. For  $P, Q \in \mathbb{P}_B$  then,  $P \leq_r Q$  if for each  $p \in P$  there exists  $q \in Q$  such that  $p \leq q$ . Furthermore, since  $Q$  is a partition,  $q$  is unique. Hence if  $f$  is a function on  $Q$ , then  $f$  can be assumed to be a function on  $P$  by defining

$$f(p) = f(q), \quad p \leq q.$$

For each  $P \in \mathbb{P}_B$ , define a complete Boolean algebra embedding  $i_P : \mathcal{P}(P) \rightarrow B$  by

$$i_P(X) = \vee X.$$

To simplify the notation,  $\mathcal{P}(P)$  is identified with the complete subalgebra of  $B$  under the embedding  $i_P$ . Clearly, if  $F$  is an (ultra-) filter on  $B$ , then  $F_P = F \cap \mathcal{P}(P)$  is an (ultra-) filter on  $\mathcal{P}(P)$ .

The ordering of ultrafilters on complete Boolean algebras will now be described in terms of the Rudin-Keisler ordering of the induced ultrafilters on the powerset subalgebras of these complete Boolean algebras.

**Definition 7.1.1 [Jipsen et al 2000]:**

Suppose that  $B$  and  $C$  are complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $B$  and  $C$  respectively.  $F \leq G$  if there exist a function  $g : \mathbb{P}_B \rightarrow \mathbb{P}_C$  and a family of functions  $\{f_P : g(P) \rightarrow P, P \in \mathbb{P}_B\}$  such that

- (a) for each  $X \subseteq P$ ,  $\vee X \in F$  iff  $\vee f_P^{-1}[X] \in G$  i.e.  $F_P \leq_{f_P} G_{g(P)}$  for each  $P \in \mathbb{P}_B$ ,
- (b) the family  $\{f_P : P \in \mathbb{P}_B\}$  satisfies the following concurrency condition for each  $P, Q \in \mathbb{P}_B$ ,

$$P \leq_r Q \text{ implies } \vee \{t \in T : f_P(t) \leq f_Q(t)\} \in G,$$

where  $T$  is the coarsest common refinement of  $g(P)$  and  $g(Q)$ .

**Remark 7.1.2 :**

To prove that the equivalence in (a) holds, it is sufficient to prove the implication from left to right since it implies its converse by virtue of the fact that the inverse function preserves complements and arbitrary joins.

To gain some insight into the connection between the definition above and the usual definition, the duality between the category of sets with functions and the category of powerset Boolean algebras with complete homomorphisms needs to be examined. If there exists a function  $f : J \rightarrow I$ , then  $\tilde{f} : \mathcal{P}(I) \rightarrow \mathcal{P}(J)$  defined by

$$\tilde{f}(X) = f^{-1}[X], \quad X \subseteq I,$$

is a complete homomorphism since  $f^{-1}$  preserves complements and arbitrary joins. Conversely, if  $h : \mathcal{P}(I) \rightarrow \mathcal{P}(J)$  is a complete homomorphism, then  $\hat{h} : J \rightarrow I$  defined by

$$\hat{h}(j) = i \text{ iff } \{j\} \subseteq h(\{i\})$$

is a function. Hence  $F \leq G$  in the Rudin-Keisler ordering iff there exists a complete homomorphism  $h : \mathcal{P}(I) \rightarrow \mathcal{P}(J)$  such that  $h[F] \subseteq G$ .

The following proposition generalises the last statement for complete Boolean algebras.

**Proposition 7.1.3 [Jipsen et al 2000] :**

*Suppose that  $B$  and  $C$  are complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $B$  and  $C$  respectively. If  $h : B \rightarrow C$  is a complete homomorphism such that  $h[F] \subseteq G$ , then  $F \leq G$ .*

**Proof :**

Define a function  $g : \mathbb{P}_B \rightarrow \mathbb{P}_C$  by

$$g(P) = h[P] - \{0\}.$$

$g$  is well defined:

The range of  $g$  consists of pairwise disjoint elements of  $C$  since  $h$  preserves meets, and  $\vee g(P) = 1$  since  $h$  preserves arbitrary joins.

Also, since  $h$  preserves meets,  $g$  is injective on partitions of which none of the elements are mapped to 0. Hence there exists an inverse function  $f_P : g(P) \rightarrow P$  defined by

$$f_P(t) = p \text{ iff } h(p) = t.$$

Suppose that  $X \subseteq P$ , and  $\vee X \in F$ . Then  $\vee f_P^{-1}[X] = \vee h[X] = h(\vee X) \in G$ . Since  $h$  is order-preserving, for  $P, Q \in \mathbb{P}_B$ , if  $P \leq_r Q$ , then  $g(P) \leq_r g(Q)$ , and for each  $t \in g(P)$ ,  $f_P(t) \leq f_Q(t)$ , and hence  $\{f_P : P \in \mathbb{P}_B\}$  satisfies the concurrency condition.  $\square$

The Rudin-Keisler ordering for complete Boolean algebras is equivalent to the usual ordering in the case that  $B$  and  $C$  are powerset Boolean algebras. One implication is immediate by Proposition 7.1.3. and the observations preceding it. Conversely, suppose that  $B = \mathcal{P}(I), C = \mathcal{P}(J)$  and there exist a function  $g : \mathbb{P}_B \rightarrow \mathbb{P}_C$  and a family of functions  $\{f_P : g(P) \rightarrow P, P \in \mathbb{P}_B\}$  such that for each  $X \subseteq I$ ,  $\vee X \in F$  iff  $\vee f_P^{-1}[X] \in G$ .

Consider the partition  $P_I = \{\{i\} : i \in I\}$ , the finest partition in  $\mathbb{P}_B$ , and the corresponding partition  $P_J$  in  $\mathbb{P}_C$ . If  $k$  is the natural isomorphism between a set and the set containing its singleton subsets, then the function  $f : J \rightarrow I$  defined by

$$f(j) = (k^{-1} \circ f_{P_I} \circ k)(j)$$

is such that for each  $X \subseteq I$ ,  $X \in F$  iff  $f^{-1}[X] \in G$  i.e.  $F \leq G$  in the usual Rudin-Keisler ordering.  $\square$

The relation  $\leq$  is a quasi-order on the class of ultrafilters on complete Boolean algebras. If  $F \leq G$  and  $G \leq F$ , then  $F$  and  $G$  are  $\leq$ -equivalent, in symbols:  $F \approx G$ . If  $F$  and  $G$  are ultrafilters on a single Boolean algebra  $B$ , then the partially ordered set of equivalence classes is denoted by  $RK(B)$ .

## 7.2 The characterisation of the Rudin-Keisler ordering by elementary embeddings between Boolean ultrapowers

Recall that a complete model  $\overline{M}$  is the model which has universe  $M$ , and in which each relation  $R$  and each function  $f$  on  $M$  is the interpretation of a relation symbol  $\overline{R}$ , and a function symbol  $\overline{f}$  respectively.

Blass [1970] proves the following characterisation theorem for the Rudin-Keisler ordering of ultrafilters on powerset Boolean algebras.

### Theorem 7.2.1 [A. Blass 1970] :

*Suppose that  $F$  and  $G$  are ultrafilters on the powerset Boolean algebras  $\mathcal{P}(I)$  and  $\mathcal{P}(J)$  respectively. The following are equivalent:*

- (a)  $F \leq G$ .
- (b) *For each model  $M$ , there exists an elementary embedding from  $M^I/F$  into  $M^J/G$ .*
- (c) *There exists an elementary embedding from  $\overline{I}^I/F$  into  $\overline{I}^J/G$ .*

### Proof:

Trivially, (b) implies (c), and hence it needs to be shown that (a) implies (b), and (c) implies (a). Suppose that there exists a function  $f : J \rightarrow I$  such that for each  $X \subseteq I$ ,  $X \in F$  iff  $f^{-1}[X] \in G$ . Define a function  $e : M^I/F \rightarrow M^J/G$  by

$$e(k/F) = (k \circ f)/G.$$

Suppose that  $R$  is a relation on  $M$ , and  $k_1, \dots, k_n \in M^I$ . Then

$$\begin{aligned} M^I/F \models R(k_1/F, \dots, k_n/F) &\text{ iff } \{i \in I : M \models R(k_1(i), \dots, k_n(i))\} \in F \\ &\text{ iff } f^{-1}[\{i \in I : M \models R(k_1(i), \dots, k_n(i))\}] \in G \\ &\text{ iff } \{j \in J : M \models R(k_1(f(j)), \dots, k_n(f(j)))\} \in G \\ &\text{ iff } M^J/G \models R((k_1 \circ f)/G, \dots, (k_n \circ f)/G) \\ &\text{ iff } M^J/G \models R(e(k_1/F), \dots, e(k_n/F)), \end{aligned}$$

and hence (a) implies (b).

Now suppose that there exists an elementary embedding  $e : \bar{I}^I/F \rightarrow \bar{I}^J/G$ . Choose  $f \in e(id_I/F)$ . Then for each  $X \subseteq I$ ,

$$\begin{aligned} X \in F &\text{ iff } \{i \in I : \bar{I} \models \bar{X}(id_I(i))\} \in F \\ &\text{ iff } \bar{I}^I/F \models \bar{X}(id_I/F) \\ &\text{ iff } \bar{I}^J/G \models \bar{X}(e(id_I/F)) \\ &\text{ iff } \{j \in J : \bar{I} \models \bar{X}(f(j))\} \in G \\ &\text{ iff } f^{-1}[X] \in G, \end{aligned}$$

and hence (c) implies (a). □

The definition of the Rudin-Keisler ordering of ultrafilters on complete Boolean algebras enable Jipsen et al [2000] to generalise Blass' theorem for Boolean ultrapowers.

**Theorem 7.2.2 [Jipsen et al 2000] :**

*Suppose that  $B$  and  $C$  are complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $B$  and  $C$  respectively. The following are equivalent:*

- (a)  $F \leq G$ .
- (b) *For each model  $M$ , there exists an elementary embedding from  $M[B]/F$  into  $M[C]/G$ .*
- (c) *There exists an elementary embedding from  $\bar{B}[B]/F$  into  $\bar{B}[C]/G$ .*

**Proof :**

As in Theorem 7.2.1, since (b) implies (c) trivially, it is sufficient to show that (a) implies (b), and (c) implies (a). Suppose that there exists a function  $g : \mathbb{P}_B \rightarrow \mathbb{P}_C$  and a family of functions  $\{f_P : g(P) \rightarrow P, P \in \mathbb{P}_B\}$  such that for each  $X \subseteq P, \forall X \in F \text{ iff } \forall f_P^{-1}[X] \in G$ . Define a function  $e : M[B]/F \rightarrow M[C]/G$  by

$$e(k/F) = (k \circ f_{\text{dom } k})/G.$$

Suppose that  $R$  is a relation on  $M$ , and  $k_1, \dots, k_n \in M[B]$  with domains  $P_1, \dots, P_n$  respectively. Then

$$M[B]/F \models R(k_1/F, \dots, k_n/F)$$

iff  $\forall \{p \in P : M \models R(k_1(p), \dots, k_n(p))\} \in F$  where  $P$  is the coarsest common refinement of  $P_1, \dots, P_n$

iff  $\forall f_P^{-1}[\{p \in P : M \models R(k_1(p), \dots, k_n(p))\}] \in G$

iff  $\forall \{q \in g(P) : M \models R(k_1(f_P(q)), \dots, k_n(f_P(q)))\} \in G$ .

Since the family  $\{f_P : P \in \mathbb{P}_B\}$  satisfies the concurrency condition and  $P \leq_R P_i$  for each  $1 \leq i \leq n$ ,

$$\forall \{q \in Q_i : f_P(q) \leq f_{Q_i}(q)\} \in G,$$

where  $Q_i$  is the coarsest common refinement of  $g(P)$  and  $g(P_i)$  i.e.

$$\forall \{q \in Q_i : k_i(f_P(q)) = k_i(f_{P_i}(q))\} \in G \text{ for each } 1 \leq i \leq n.$$

Then

$$\forall \{q \in Q : k_i(f_P(q)) = k_i(f_{P_i}(q)) \text{ for each } 1 \leq i \leq n\} \in G,$$

where  $Q$  is the coarsest common refinement of  $Q_1, \dots, Q_n$ . Hence

$$\begin{aligned} & \forall \{q \in g(P) : M \models R(k_1(f_P(q)), \dots, k_n(f_P(q)))\} \in G \\ \text{iff } & \forall \{q \in Q : M \models R(k_1(f_{P_1}(q)), \dots, k_n(f_{P_n}(q)))\} \in G \\ \text{iff } & M[C]/G \models R(k_1 \circ f_{P_1}/G, \dots, (k_n \circ f_{P_n})/G) \\ \text{iff } & M[C]/G \models R(e(k_1/F), \dots, e(k_n/F)), \end{aligned}$$

and (a) implies (b).

Now suppose that there exists an elementary embedding  $e : \overline{B}[B]/F \rightarrow \overline{B}[C]/G$ . For each  $P \in \mathbb{P}_B$ , choose  $f_P \in e(id_P/F)$ , and define  $g(P) = \text{dom } f_P$ . Although  $\text{range } f_P \subseteq \overline{B}$ , it can be assumed that  $\text{range } f_P \subseteq P$ :

Suppose that  $\overline{P}$  is a relation on  $\overline{B}$  such that  $\overline{B} \models \overline{P}(p)$  iff  $p \in P$ . Since  $\forall \{p \in P : \overline{B} \models \overline{P}(id_P(p))\} = 1 \in F$ ,  $\overline{B}[B]/F \models \overline{P}(id_P/F)$  and hence  $\overline{B}[C]/G \models \overline{P}(e(id_P/F))$ . This implies that  $\overline{B}[C]/G \models \overline{P}(f_P/G)$  i.e.  $\forall \{q \in g(P) : \overline{B} \models \overline{P}(f_P(q))\} \in G$ . Suppose that  $\forall \{q \in g(P) : \overline{B} \models \overline{P}(f_P(q))\} = c$ . Hence if  $q \leq c$ , then  $f_P(q) \in P$ . Fix  $\tilde{p} \in P$ , and define  $\tilde{f}_P : g(P) \rightarrow P$  by

$$\tilde{f}_P(q) = \begin{cases} f_P(q), & q \leq c \\ \tilde{p}, & \text{otherwise.} \end{cases}$$



Since  $\vee \{q \in g(P) : \tilde{f}_P(q) = f_P(q)\} \in G$ ,  $\tilde{f}_P/G = f_P/G$ . Suppose that  $\bar{X}$  is a relation on  $\bar{B}$  such that  $\bar{B} \models \bar{X}(p)$  iff  $p \in X$ . Then

$$\begin{aligned}
 \vee X \in F &\text{ iff } \vee \{p \in P : \bar{B} \models \bar{X}(id_P(p))\} \in F \\
 &\text{ iff } \bar{B}[B]/F \models \bar{X}(id_P/F) \\
 &\text{ iff } \bar{B}[C]/G \models \bar{X}(e(id_P/F)) \\
 &\text{ iff } \bar{B}[C]/G \models \bar{X}(f_P/G) \\
 &\text{ iff } \vee \{q \in g(P) : \bar{B} \models \bar{X}(f_P(q))\} \in G \\
 &\text{ iff } \vee f_P^{-1}[\{p \in P : \bar{B} \models \bar{X}(p)\}] \in G \\
 &\text{ iff } \vee f_P^{-1}[X] \in G.
 \end{aligned}$$

In order to prove the concurrency condition, suppose that  $P \leq_r Q$ , and  $\bar{R}$  is a relation on  $\bar{B}$  such that  $\bar{B} \models \bar{R}(p, q)$  iff  $p \in P, q \in Q$  and  $p \leq q$ . Now  $\vee \{p \in P : \bar{B} \models \bar{R}(id_P(p), id_Q(p))\} = 1$ . Hence  $\bar{B}[B]/F \models \bar{R}(id_P/F, e(id_Q/F))$ , and hence  $\bar{B}[C]/G \models \bar{R}(e(id_P/F), e(id_Q/F))$  i.e.  $\bar{B}[C]/G \models \bar{R}(f_P/G, f_Q/G)$ . This implies that  $\vee \{t \in T : \bar{B} \models \bar{R}(f_P(t), f_Q(t))\} \in G$  where  $T$  is the coarsest common refinement of  $g(P)$  and  $g(Q)$ , which is equivalent to the concurrency condition.  $\square$

**Remark 7.2.3 :**

In the definition of  $F \leq G$ , it is sufficient to consider partitions on a dense subsemilattice  $S$  of  $\mathbb{P}_B$ , since if  $f \in M[B]$ , it can be assumed that  $dom f \in S$ .

Recall that the relative algebra  $B \upharpoonright b$  of a Boolean algebra  $B$  w.r.t.  $b \in B$  (Definition 0.1.27) is the Boolean algebra  $B \upharpoonright b = \{x \in B : x \leq b\}$ , in which the partial ordering is inherited from  $B$ .

**Example 7.2.4 :**

Suppose that  $\{B_i : i \in I\}$  is a family of complete Boolean algebras. For each  $i \in I$ , suppose that  $\langle b \rangle_i$  is the element of  $\Pi\{B_i : i \in I\}$  such that  $\langle b \rangle_i(i) = 1_{B_i}$  and  $\langle b \rangle_i(j) = 0_{B_j}$  for each  $j \neq i$ . Then  $\{\langle b \rangle_i : i \in I\}$  is a partition of  $\Pi\{B_i : i \in I\}$ . Hence  $B_i$  can be isomorphically embedded into the relative algebra  $\Pi\{B_i : i \in I\} \upharpoonright \langle b \rangle_i$  for each  $i$ . Suppose that  $e_i : B_i \rightarrow \Pi\{B_i : i \in I\}$  is this relative embedding.  $\pi_i \circ e_i$  is the identity on  $B_i$ , and  $e_i$  preserves all existing joins and meets. Consider a family of partitions  $\{P_i \in \mathbb{P}_{B_i} : i \in I\}$ . Then  $\cup\{e_i[P_i] : i \in I\}$  is a partition of  $\Pi\{B_i : i \in I\}$  and the set of all such partitions is a dense subsemilattice of  $\Pi\{B_i : i \in I\}$ .

If  $F$  is an ultrafilter on  $B$  and  $b \in F$ , denote the set  $\{x \wedge b : x \in F\}$  by  $F \upharpoonright b$ . Note that  $F \upharpoonright b$  is an ultrafilter on  $B \upharpoonright b$ . The following result is obtained by resorting to the Characterisation Theorem.

**Proposition 7.2.5 [Jipsen et al 2000] :**

*Suppose that  $B$  and  $C$  are complete Boolean algebras, and  $F$  and  $G$  are ultrafilters on  $B$  and  $C$  respectively. The following are equivalent:*

- (a)  $F \leq G$ .
- (b) *There exist  $b \in F$  and  $c \in G$  such that  $F \upharpoonright b \leq G \upharpoonright c$ .*
- (c) *There exist  $b \in F$  and a complete subalgebra  $\tilde{C}$  of  $C$  such that  $F \upharpoonright b \leq \tilde{C} \cap G$ .*

**Proof :**

Clearly, (a) implies (b) if  $b = 1_B$  and  $c = 1_C$ . Similarly, (a) implies (c) if  $b = 1_B$  and  $\tilde{C} = C$ . Suppose that (b) holds. Now for any model  $M$ ,  $M[B]/F \cong M[B \upharpoonright b]/F \upharpoonright b$ . Since  $F \upharpoonright b \leq G \upharpoonright c$ , by Theorem 7.2.2,  $M[B \upharpoonright b]/F \upharpoonright b$  is elementarily embeddable into  $M[C \upharpoonright c]/G \upharpoonright c$ , and since  $M[C \upharpoonright c]/G \upharpoonright c \cong M[C]/G$ , it follows that  $M[B]/F$  is elementarily embeddable into  $M[C]/G$  i.e.  $F \leq G$  by Theorem 7.2.2. Hence (b) implies (a). Similarly, it can be shown that (c) implies (a) by observing that  $M[\tilde{C}]/\tilde{C} \cap G$  is elementarily embeddable into  $M[C]/G$ .  $\square$

### 7.3 The conditions under which the Rudin-Keisler poset of a Boolean algebra is order-embeddable into the Rudin-Keisler poset of another Boolean algebra

The case of relative algebras will first be considered.

**Lemma 7.3.1 [Jipsen et al 2000] :**

*Suppose that  $B$  is a Boolean algebra and  $C = B \upharpoonright b$  is a relative algebra of  $B$ . If  $F$  is an ultrafilter on  $C$ , then  $\tilde{F} = \{x \in B : x \geq y \text{ for some } y \in F\}$  is an ultrafilter on  $B$ .*

**Proof :**

It is easy to check that  $\tilde{F}$  is a filter. Suppose that  $x \in B$ . Then  $x \wedge b \in C$ , and hence  $x \wedge b \in F$  or  $(x \wedge b)^c \wedge b \in F$ . Since  $(x \wedge b)^c \wedge b = x^c \wedge b$ , it follows that either  $x \in \tilde{F}$  or  $x^c \in \tilde{F}$ .  $\square$

**Corollary 7.3.2 [Jipsen et al 2000] :**

*If  $C$  is isomorphic to a relative algebra of  $B$ , then  $RK(C)$  is order-embeddable into  $RK(B)$ .*

**Proof :**

It can be assumed that there exists  $b \in B$  such that  $C = B \upharpoonright b$ . Suppose that  $F$  and  $G$  are ultrafilters on  $C$ . Then  $F = \tilde{F} \upharpoonright b$  and  $G = \tilde{G} \upharpoonright b$ . By Proposition 7.2.5, it follows that  $\tilde{F} \leq \tilde{G}$  iff  $F \leq G$ .  $\square$

Next, the case of direct powers of complete Boolean algebras is considered.

Suppose that  $J$  is a set and  $B$  is a complete Boolean algebra, and consider the direct power  $B^J$ . If  $F$  is an ultrafilter on  $B$ , and  $H$  is an ultrafilter on  $\mathcal{P}(J)$ , define  $F_H$  by

$$F_H = \{k \in B^J : k^{-1}[F] \in H\}.$$

If  $B = \mathcal{P}(I)$ , then  $B^J$  is isomorphic to  $\mathcal{P}(I \times J)$  and  $F_H$  is isomorphic to the ultrafilter  $F \otimes G$  as given in the introduction in Chapter 2.

**Lemma 7.3.3 [Jipsen et al 2000] :**

*Suppose that  $B$  is a complete Boolean algebra, and  $F, G$  and  $H$  are ultrafilters on  $B, \mathcal{P}(I)$  and  $\mathcal{P}(J)$  respectively. If  $G \leq H$ , then  $F_G \leq F_H$ .*

**Proof :**

Suppose that  $G \leq H$ . Then there exists a function  $f : J \rightarrow I$  such that for each  $X \subseteq I$ ,  $X \in G$  iff  $f^{-1}[X] \in H$ . Suppose that  $k \in B^I$  and define a function  $h : B^I \rightarrow B^J$  by

$$h(k) = k \circ f.$$

Since the operations on  $B^I$  are defined pointwise,  $h$  is a complete homomorphism. If  $k \in F_G$ , then  $k^{-1}[F] \in G$ , and hence  $(k \circ f)^{-1}[F] = f^{-1}[k^{-1}[F]] \in H$ .  $\square$

The converse of Lemma 7.3.3 is not as straightforward and requires a further assumption. The proof requires the application of the well-known result that if  $F$  is  $\kappa$ -complete and  $P \in \mathbb{P}_B^\kappa$ , then  $F$  and  $P$  have exactly one element in common. (See Jipsen and Rose [1999], Proposition 0.9).

**Lemma 7.3.4 [Jipsen et al 2000]:**

*Suppose that  $B$  is a complete Boolean algebra, and  $F, G$  and  $H$  are ultrafilters on  $B, \mathcal{P}(I)$  and  $\mathcal{P}(J)$  respectively. If  $F$  is  $|I|^+$ -complete and  $F_G \leq F_H$ , then  $G \leq H$ .*

**Proof :**

Suppose that  $F_G \leq F_H$ . Then there exist a function  $g : \mathbb{P}_{B^I} \rightarrow \mathbb{P}_{B^J}$  and a family of functions  $\{f_P : g(P) \rightarrow P, P \in \mathbb{P}_{B^I}\}$  such that for each  $X \subseteq P$ ,  $\forall X \in F_G$  iff  $f_P^{-1}[X] \in F_H$ . Suppose that  $\chi_K$  is the characteristic function of  $K \subseteq I$  or  $K \subseteq J$ , and consider the partition  $P_I = \{\chi_{\{i\}} \in B^I : i \in I\}$  in  $B^I$  and the corresponding partition  $P_J$  in  $B^J$ . Suppose that  $h : \mathcal{P}(P_I) \rightarrow \mathcal{P}(g(P_I))$  is a complete homomorphism and is defined by

$$h(\vee X) = \vee f_{P_I}^{-1}[X].$$

Define a function  $f : J \rightarrow I$  by

$$f(j) = i \quad \text{iff} \quad \pi_j(h(\chi_{\{i\}})) \in F.$$

$f$  is well defined, since  $F$  is  $|I|^+$ -complete and hence  $F$  and the partition  $(\pi_j \circ h)[P_I] - \{0\}$  have exactly one element in common. Now suppose that  $X \in G$ . Then  $\{i \in I : \chi_X(i) \in F\} \in G$ , and hence  $\chi_X \in F_G$ . Since  $\chi_X = \vee \{\chi_{\{i\}} : i \in X\}$ ,  $h(\chi_X) = \vee \{f_{P_I}^{-1}(\chi_{\{i\}}) : i \in X\}$  by definition of  $h$ , and hence  $h(\chi_X) \in F_H$  i.e.  $(h(\chi_X))^{-1}[F] \in H$ .

Now

$$\begin{aligned} j \in (h(\chi_X))^{-1}[F] &\text{ iff } h(\chi_X)(j) \in F \\ &\text{ iff } \vee \{h(\chi_{\{i\}})(j) : i \in X\} \in F \text{ since} \\ &\chi_X = \vee \{\chi_{\{i\}} : i \in X\} \\ &\text{ iff } h(\chi_{\{i\}})(j) \in F \text{ for some } i \in X \text{ since} \\ &F \text{ is } |I|^+ \text{-complete} \\ &\text{ iff } f(j) \in X \text{ since } h(\chi_{\{i\}})(j) = \pi_j(h(\chi_{\{i\}})) \\ &\text{ iff } j \in f^{-1}[X], \end{aligned}$$

and the result follows. □

The following theorem follows immediately from Lemma 7.3.3 and Lemma 7.3.4.

**Theorem 7.3.5 [Jipsen et al 2000] :**

*Suppose that  $B$  is a complete Boolean algebra, and suppose that there exists a  $\kappa^+$ -complete ultrafilter on  $B$ . Then the poset  $RK(\mathcal{P}(\lambda))$  is order-embeddable into the poset  $RK(B^\kappa)$  for each  $\lambda \leq \kappa$ .*

If  $B$  is homogeneous and  $B$  has a partition of cardinality  $\kappa$ , then  $B^\kappa \cong B$ . Furthermore, if there exists a  $\kappa^+$ -complete ultrafilter on  $B$ , then  $RK(\mathcal{P}(\kappa))$  is order-embeddable into  $RK(B)$  by the above theorem.

**Problems 7.3.6 :**

- (a) Can Theorem 7.3.5 be proved in *ZFC* i.e. without the assumption that a  $\kappa^+$ -ultrafilter exists on  $B$ ?
  - (b) Suppose that  $A$  and  $B$  are complete Boolean algebras, and  $A$  is a regular subalgebra of  $B$ . Is it true that  $RK(A)$  is order-embeddable into  $RK(B)$ ? In particular, if  $B$  has a partition of cardinality  $\kappa$ , is it true that  $RK(\mathcal{P}(\omega))$  is order-embeddable into  $B$ ?
-

## List of Symbols

$\subseteq$	is a subset (or submodel) of
$\models$	satisfies
$\cong$	is isomorphic to
$\equiv$	is elementarily equivalent to
$\prec$	is an elementary submodel of
$\in$	is an element of
$\emptyset$	the empty set
$\leq$	is less than or equal to in the partial ordering
$\leq_r$	is a refinement of
$\sim$	is equivalent to
$\exists$	there exists
$\forall$	for all
$!$	unique
$\neg$	the negation of
$\wedge$	the meet of (or logical and)
$\vee$	the join of (or logical or)
$^c$	the complement of
$\cup$	the union of
$\cap$	the intersection of
$f : M \rightarrow N$	$f$ is a function from $M$ into $N$
$f : M \xrightarrow{P} N$	$f$ is a partial function from $M$ into $N$
$\text{dom} f$	the domain of $f$
$\text{dom}_c f$	the domain of $f$ on which it is continuous
$f \upharpoonright X$	the restriction of $f$ to $X$
$f^{-1}[X]$	the inverse image of $X$ under $f$
$f \circ g$	the composition of $f$ and $g$
$\pi_j$	the projection onto the $j$ th coordinate
$\mathcal{P}(X)$	the powerset Boolean algebra on $X$
$\mathcal{P}_\omega(\lambda)$	the set of all finite subsets of $\lambda$
$CL(X)$	the algebra of clopen subsets of $X$
$X - Y$	the relative complement of $Y$ w.r.t. $X$
$M[B]$	the Boolean power of $M$ w.r.t. $B$
$M[B]_\omega$	the bounded Boolean power of $M$ w.r.t. $B$
$M[B]_\kappa$	the $\kappa$ -bounded Boolean power of $M$ w.r.t. $B$
$M[B]/F$	the Boolean ultrapower of $M$ w.r.t. $B$ and $F$
$M[B]_\kappa/F$	the $\kappa$ -bounded Boolean ultrapower of $M$ w.r.t. $B$ and $F$
$\overline{Y}^o$	the interior of the closure of $Y$
$RO(X)$	the algebra of regular open subsets of $X$
$ULT(B)$	the set of all ultrafilters on $B$

$BS$	the category of Boolean spaces with continuous functions
$BA$	the category of Boolean algebras with homomorphisms
$C(X, M)$	the algebra of continuous functions $f : X \rightarrow M$ where $X$ has been given the discrete topology
$D(X, M)$	the algebra of functions from $X$ into $M$ where $X$ is given the discrete topology
$X/\sim$	the set of equivalence classes of $X$ w.r.t. $\sim$
$f/F$	the equivalence class of $f$ w.r.t. $F$
$\mathbb{P}_B$	the semilattice of partitions of $B$
$\mathbb{P}_B(P)$	the subsemilattice of $\mathbb{P}_B$ generated by $P$
$\mathbb{P}_B^\kappa$	the subsemilattice of $\mathbb{P}_B$ consisting of partitions with fewer than $\kappa$ elements
$B^+$	the set of non-zero elements of $B$
$M \times N$	the direct product of $M$ and $N$
$\nabla_M$	$M \times M$
$\Delta_M$	the identity relation on $M$
$\prod\{M_P : P \in \mathcal{U}\}$	the direct product of the $M_P, P \in \mathcal{U}$
$M^I$	the direct power of $M$ w.r.t. $I$
$B \oplus \tilde{B}$	the free product of $B$ and $\tilde{B}$
$B \upharpoonright p$	the relative algebra of $B$ w.r.t. $p$
$\ \varphi\ $	the Boolean value of $\varphi$
$\ \varphi\ _{M[B]}$	the Boolean value of $\varphi$ w.r.t. $M[B]$
$f = \Sigma\{b_i \cdot f_i : i \in I\}$	$\ f = f_i\  \geq b_i$ for each $i, b_i \neq b_j$ for $i \neq j$ , and $\vee \{b_i : i \in I\} = 1$
$Th(\mathcal{K})$	the theory of $\mathcal{K}$
$cf X$	the cofinality of the poset $X$
$ X $	the cardinality of $X$
$\alpha^+$	the successor of cardinal $\alpha$
$Fr_\kappa$	the free Boolean algebra on $\kappa$ generators
$0^\sharp$	zero-sharp
$Def(Y)$	the class of sets definable over $Y$
$L_\alpha$	the $\alpha$ -th hierarchy of constructible sets
$\mathcal{L}$	the class of all constructible sets
$O_n$	the class of all ordinals
$LCH$	Large Cardinal Hypothesis
$ZFC$	Zermelo-Fraenkel set theory with the Axiom of Choice
$F \otimes G$	the product ultrafilter of $F$ and $G$ , where $F$ and $G$ are ultrafilters on powerset Boolean algebras (or the product ultrafilter of $F$ and $G$ , where $F$ and $G$ are ultrafilters on complete Boolean algebras)
$A[B]$	the Boolean power of $A$ w.r.t. $B$ with the two-valued equality relation
$RK(B)$	the Rudin-Keisler poset of ultrafilters on $B$

# Index

## A

algebra, 4  
      $\kappa$ -Suslin, 70  
      $\langle m_1, m_2 \rangle$ -irreducible, 100  
     regular open, 5  
     relative, 9  
     simple, 100  
     subdirectly irreducible, 100  
 Arens, R. F., v  
 Ash, C. J., vi, 23

## B

Banaschewski, B., vi, 10  
 Benda, M., i, 16  
 Blass, A., 107  
 Blass' Characterisation Theorem, 106  
 Boole, G., 4  
 Boolean algebra, 4  
     complete, 6  
     completion of, 7  
      $\kappa$ -complete, 6  
      $\kappa$ -disjointable, 82  
      $\kappa$ -partition complete, 70  
 Boolean power, 10  
     bounded, 10  
      $\kappa$ -bounded, 21  
 Boolean space, 10  
 Boolean ultrapower, 41  
      $\kappa$ -bounded, 41  
 Boolean value, 30  
 $(B, P)$ -product of models, 45  
 $(B, P)$ -ultraproduct of models, 46  
 Burris, S., i, 4

## C

Chang, C. and Keisler, H., vii, 59  
 cofinality, 48  
 component, 11  
 common domain, 22  
 countable chain condition, 51



**D**

direct limit, 19

**E**

elementarily equivalent, 3

embedding, 3

    elementary, 3

    strong, 98

extension, 2

    complete, 89

    elementary, 3

**F**

factor sentence, 28

Feferman-Vaught Theorem, 38

filter, 7

filtral power, 41

$\kappa$ -bounded, 41

finite intersection property, 8

Finite Iteration Theorem, 69

Foster, A. L., 10

Frayne, T., i, 22

function, 80

    disjoint, 82

    monotonically decreasing, 80

    multiplicative, 80

**G**

Gelfand, I., v

Generalised Continuum Hypothesis, vi

**H**

Hall, P., v

Hanf, W., 27

Hodges, W., v

homomorphism, 2

    complete, 7

$\kappa$ -complete, 7

Huntington, E., 4

## I

inner product, 75  
isomorphic, 2

## J

Jech, T., vii  
Jeffers, E., i, 100  
Jipsen, P., i, 103  
Jónsson, B., 27

## K

$\kappa$ -chain condition w.r.t.  $B$ , 61  
 $\kappa$ -partition cofinality property, 19  
 $(\kappa, \lambda)$ -distributive law, 60  
 $(\kappa, \infty)$ -ditributive law, 61  
Kaplansky, I., v  
Keisler, H., 80  
Keisler-Shelah Theorem, 74  
Kochen, S. B., 97  
Kochen's Limit Ultrapower Theorem, 97  
Koppelberg, B., i, 48  
Koppelberg, S., i, 48  
Kunen, K., 54

## L

Lange, W., 48  
limit Boolean ultrapower, 95  
Los's Ultraproduct Theorem, 41

## M

Mansfield, R., i, 29  
model, 2  
    admits glueing over  $B$ , 37  
     $\mathcal{B}$ -separating, 27  
    complete, 96  
    full, 37  
     $\kappa$ -saturated, 74  
    of type  $(\mu, \nu)$ , 2  
    weakly separating, 28  
Morel, A., i, 22

## N

Nelson, E., vi, 10

## O

Ouwehand, P., i, 17

## P

partition, 8

Pinus, A., i, 103

Potthoff, K., ix, 89

Prikry, K., 54

## R

refinement, 17

    coarsest common, 17

Rose, H., i, 17

Rudin-Keisler ordering, 50

    of ultrafilters on powerset Boolean algebras, 50

    of ultrafilters on complete Boolean algebras, 104

## S

Scott, D., 22

set, 5

    clopen, 5

    compact, 10

    constructible, 55

    definable, 55

simplicity sentence, 100

Stone, M. H., v, 13

Stone space, 12

subalgebra, 6

    dense, 6

    independent, 48

    regular, 6

subdirect irreducibility sentence, 100

submodel, 2

    elementary, 3

subspace, 11

    connected, 11

    totally disconnected, 11

**T**

Tarski, A., 27

Tarski-Vaught Criterion, 4

Taylor, W., 100

**U**

ultrafilter, 8

$\delta$ -decomposable, 50

    descendingly countably incomplete, 74

$\kappa$ -good (Mansfield), 80

$\kappa$ -good (Benda), 85

$(\tau, \lambda)$ -regular, 53

    uniform, 54

    updirected family of, 19

**Z**

Zorn's Lemma, 8

---

## References

- Ash, C.J., *Reduced powers and Boolean extensions*, Journal of the London Mathematical Society, vol. 9 (1975), pp. 429-432.
- Banaschewski, B. and Nelson, E., *Boolean powers as algebras of continuous functions*, Dissertationes Mathematica, no. 179, Warsaw, 1980.
- Bell, J.L. and Slomson, A.B., *Models and Ultraproducts: an introduction*, North-Holland, Amsterdam-London, 1971.
- Benda, M., *Note on Boolean ultrapowers*, Proceedings of the American Mathematical Society, vol. 46, no. 2 (1974), pp. 289-293.
- Blass, A., *Orderings of Ultrafilters*, Ph.D. thesis, Harvard University, 1970.
- Burris, S., *Boolean powers*, Algebra Universalis, vol. 5 (1975), pp. 341-360.
- Burris, S. and Jeffers, E., *On the simplicity and subdirect irreducibility of Boolean ultrapowers*, Colloquium Mathematicum, vol. 39 (1978), pp. 215-218.
- Burris, S. and Sankappanavar, H., *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.
- Burris, S., *Boolean Constructions*, Universal Algebra and lattice theory proceedings, Puebla 1982, Lecture Notes in Mathematics, no. 1004, Springer-Verlag, 1983, pp. 67-90.
- Chang, C.C. and Keisler, H.J., *Model Theory*, Studies in Logic and the Foundations of Mathematics, no. 73, North-Holland, Amsterdam - New York - Oxford, 1973.
- Foster, A.L., *Generalised "Boolean" theory of universal algebras*, part 1, Mathematische Zeitung, vol. 58 (1953), pp. 306-336.
- Frayne, T., Morel, A. and Scott, D., *Reduced direct products*, Fundamenta Mathematicae, vol. 51 (1962), pp. 195-228.
- Garavaglia, S. and Plotkin, J.M., *Separation properties and Boolean powers*, Colloquium Mathematicum, vol. 48 (1984), pp. 167-172.
- Hodges, W., *Model Theory*, Encyclopaedia of Mathematics and its Applications, no. 42, Cambridge University Press, 1993.
- Jech, T., *Set Theory*, Academic Press, New York, 1978.
- Jipsen, P. and Rose, H., *Partition complete Boolean algebras and almost compact cardinals*, Mathematical Logic Quarterly, vol. 45, no. 2 (1999), pp. 241-255.

- Jipsen, P., Pinus, A. and Rose, H., *Rudin-Keisler posets of complete Boolean algebras*, preprint (2000), Mathematical Logic Quarterly (to appear).
- Koppelberg, B. and Koppelberg, S., *A Boolean ultrapower which is not an ultrapower*, Journal of Symbolic Logic, vol. 41, no. 1 (1976), pp. 245-249.
- Koppelberg, B., *Ultrapowers and Boolean ultrapowers of  $\omega$  and  $\omega_1$* , Archiv für Mathematische Logik und Grundlagenforschung, vol. 20 (1980), pp. 147-153.
- Kunen, K. and Prikry, K., *On descendingly incomplete ultrafilters*, Journal of Symbolic Logic, vol. 36 (1971), pp. 650-653.
- Mansfield, R., *The theory of Boolean ultrapowers*, Annals of Mathematical Logic, vol. 2 (1971), pp. 297-325.
- Monk, J.D. and Bonnet, R. (editors), *Handbook of Boolean algebras*, vol. 1-3, North-Holland, Amsterdam - New York - Oxford, 1989.
- Ouwehand, P. and Rose, H., *Filtral powers of structures*, Journal of Symbolic Logic, vol. 63, no. 4 (1998), pp. 1239-1254.
- Pothoff, K., *Boolean ultrapowers*, Archiv für Mathematische Logik und Grundlagenforschung, vol. 16 (1974), pp. 37-48.
- Quackenbush, R.W., *Free products of bounded distributive lattices*, Algebra Universalis, vol. 2 (1972), pp. 393-394.
- Taylor, W., *Residually small varieties*, Algebra Universalis, vol. 2 (1972), pp. 33-53.
-